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SOLVABILITY AND CONSISTENCY FOR LINEAR EQUATIONS AND INEQUALITIES*

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1. Introduction. The term *consistent* is in common use in two contexts that appear to be quite different at first sight. It is often applied to systems of linear equations as a synonym for *solvable*. Thus, Dickson says in [3]: "We shall call two or more equations *consistent* if there exist values of the unknowns which satisfy all of the equations." Again, Bôcher writes in [1]: "The equations may have no solution, in which case they are said to be *inconsistent*." Elsewhere, it is applied by logicians to deductive systems as a synonym for *non-contradictory*. Thus, Tarski defines the term in [8]: "A deductive theory is called *consistent* or non-contradictory if no two asserted statements of this theory contradict each other."

Our preliminary purpose is to reconcile these two usages, agreeing informally that "solvable" means "satisfiable" and that "consistent" means "non-contradictory." The reconciliation is brought about by setting forth explicitly the definition of consistency that has been employed implicitly in ordinary treatments of linear equations. This definition is based on a non-effective characterization of the logical consequences of the system, and is almost trivially equivalent to solvability under much more general conditions than are considered here. Therefore, this equivalence adds little or nothing to our knowledge of the special subject of linear equations.

These considerations are in sharp contrast with another use of consistency, which considers only those consequences that can be derived by applying a finite number of algebraic operations to the system. We shall show that the ordinary criteria for the solvability of systems of linear equations follow directly from the latter notion when the only algebraic operation that is allowed is that of forming linear combinations. Whatever novelty there is in this approach consists in viewing the system of equations as a set of postulates added to an underlying logic that includes the laws of real numbers, and then investigating the "methods of proof" appropriate to it.

The main object of this paper is to extend this formulation to systems of linear inequalities. It is somewhat remarkable that the same theorems persist with only minor modifications, namely, with reasonable care in the type of linear combinations that are allowed, and with the use of a process of elimination designed for linear inequalities. The criteria for solvability that are proved in this manner form a basis for the modern disciplines of linear programming and game theory. They also admit important geometric interpretations. These and other applications will be treated in a sequel.

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2. Systems of linear equations. The system of linear equations

$$\begin{aligned} 3x - 6y + 4z &= 1 \\ -x + 2y - 2z &= 3 \\ x - 2y + z &= 0 \end{aligned}$$

does not have a solution. Despite its triviality, let us examine an argument that might be advanced as a proof of this statement.

Suppose the equations are multiplied by new unknowns, u, v, and w, respectively, and added. Represent this by listing the multipliers to the left of the equations and the sum below, thus:

$$\begin{array}{lccc} u: & 3x & -6y & +4z = 1 \\ v: & -x & +2y & -2z = 3 \\ w: & x & -2y & +z = 0 \\ \hline & (3u - v + w)x + (-6u + 2v - 2w)y + (4u - 2v + w)z = u + 3v & & \end{array}$$

If the original system were satisfied by numbers $\bar{x}$, $\bar{y}$, and $\bar{z}$, then the same operations would yield

$$(3u - v + w)\bar{x} + (-6u + 2v - 2w)\bar{y} + (4u - 2v + w)\bar{z} = u + 3v$$

and this would be a true statement for all values of u, v, and w. However, if the multipliers $u = 1$, $v = 1$, $w = -2$ are chosen, this says

$$0\bar{x} + 0\bar{y} + 0\bar{z} = 4,$$

a false statement about numbers.

We shall see later how such multipliers can be found by successively eliminating the unknowns. For the moment it is important to remark that they were chosen to satisfy

$$\begin{aligned} 3u - v + w &= 0 \\ -6u + 2v - 2w &= 0 \\ 4u - 2v + w &= 0 \\ u + 3v &\neq 0 \end{aligned}$$

and thus establish the contradiction. This is, of course, the original system transposed in a special way.

So much for motivation; the object of this section is to state precise conditions for the solvability of general *systems* S of linear equations. The systems will be assumed to contain r inhomogeneous equations in n unknowns, $x_1, \cdots, x_n$, and hence will have the form

$$
\begin{aligned}
&c_{11}x_1 + \cdots + c_{1n}x_n = c_1 \\
&\qquad\qquad \cdots\cdots\cdots\cdots \\
&c_{r1}x_1 + \cdots + c_{rn}x_n = c_r.
\end{aligned}
\tag{S}
$$

The c_{kl} and c_k are given real numbers (for $k=1, \cdots, r$ and $l=1, \cdots, n$) and define the system S. The adjective "inhomogeneous" means that no assumption is made about the right hand members c_k.

An indexed set $\overline{X}$ of real numbers $(\bar{x}_1, \cdots, \bar{x}_n)$ is called a *solution* for S if all of the equations

$$c_{k1}\bar{x}_1 + \cdots + c_{kn}\bar{x}_n = c_k \qquad (k = 1, \cdots, r)$$

are true statements. The set of solutions for S is denoted by $\mathfrak{S}(S)$ and S is called *solvable* if $\mathfrak{S}(S)$ is not empty. An equation

$$d_1x_1 + \cdots + d_nx_n = d$$

in the unknowns $x_1, \cdots, x_n$ is called a (logical) *consequence* of S if $d_1\bar{x}_1 + \cdots + d_n\bar{x}_n = d$ is a true equation whenever $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n)$ is a member of $\mathfrak{S}(S)$, in symbols, if $\overline{X} \in \mathfrak{S}(S)$. A curious situation occurs if S is not solvable, Then the definition of a consequence places no restriction on an equation $d_1x_1 + \cdots + d_nx_n = d$ because there are no solutions $\bar{x}_1, \cdots, \bar{x}_n$ to check for equality. Hence we are forced to conclude that, if S is not solvable, then *every* equation in the unknowns $x_1, \cdots, x_n$ is a consequence of S.

Our first definition of consistency will be based on the idea of consequence and the form of the patently false equation, $0\bar{x} + 0\bar{y} + 0\bar{z} = 4$, derived in the example above. Namely, a system S is said to be *inconsistent* if some equation

$$0x_1 + \cdots + 0x_n = d, \qquad \text{with } d \neq 0,$$

is a consequence of S; otherwise it is called *consistent*. (Thus, the contradictory assertions used in this definition are $0 = d$ and $d \neq 0$.)

The first theorem will establish the logical connection between solvability and this notion of consistency. Its statement explains immediately why these terms have often been confused (or used interchangeably) in textbook discussions of linear equations. The triviality of its proof reveals that it is a theorem without special content for linear equations.

THEOREM 1. *A system S is solvable if and only if it is consistent.*

Proof. If S is solvable, choose an $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n) \in \mathfrak{S}(S)$. Then $0\bar{x}_1 + \cdots + 0\bar{x}_n$ is equal to zero by the rules of operating with real numbers and hence $0\bar{x}_1 + \cdots + 0\bar{x}_n = d$ is not a true equation for any $d \neq 0$. Therefore, no $0x_1 + \cdots + 0x_n = d$, with $d \neq 0$, is a consequence of S, and S is consistent.

On the other hand, if S is not solvable, then every equation in $x_1, \cdots, x_n$ is a consequence of S. In particular, $0x_1 + \cdots + 0x_n = 1$ is a consequence of S, and S is inconsistent.

The significant content of the theory of linear equations comes not from the fact that an absurd consequence can always be exhibited when S is not solvable, but from the *form* of this equation and from *how* it can be obtained. To this end, form a scheme analogous to that used in the example, with multipliers appearing at the left of the system and the sum appearing below:

$$\begin{aligned} w_1&: c_{11}x_1 + \cdots + c_{1n}x_n = c_1 \\ w_2&: c_{21}x_1 + \cdots + c_{2n}x_n = c_2 \\ &\cdots\cdots\cdots\cdots \\ w_r&: c_{r1}x_1 + \cdots + c_{rn}x_n = c_r \\ \hline &d_1x_1 + \cdots + d_nx_n = d. \end{aligned}$$

The coefficients of the sum are easily read off; they are:

$$\begin{aligned} d_1 &= w_1c_{11} + \cdots + w_rc_{r1} \\ &\cdots\cdots\cdots\cdots \\ d_n &= w_1c_{1n} + \cdots + w_rc_{rn} \\ d &= w_1c_1 + \cdots + w_rc_r. \end{aligned}$$

An equation,

$$d_1x_1 + \cdots + d_nx_n = d,$$

that is formed in this manner, is called a *linear combination* of the equations of S. The multipliers $w_1, \cdots, w_r$ are called the *coefficients* of the linear combination.

With this set of definitions, the central theorem on the solvability of systems of inhomogeneous linear equations is:

THEOREM 2. (a) *Every linear combination of the equations of S is a consequence of S.* (b) *If S is solvable, then every consequence of S is a linear combination of the equations of S. If S is not solvable, then an equation*

$$0x_1 + \cdots + 0x_n = d, \qquad \text{with } d \neq 0,$$

is a linear combination of the equations of S.

Several informal remarks may help to explain the content and intent of this theorem. First, it characterizes the consequences of *solvable* systems S as being exactly the linear combinations. Since "consequence" is a logical notion and "linear combination" is algebraic, this is quite remarkable. Often we may obtain the information that some statement holds whenever other statements are true; however, it is seldom that this forces such a narrow and explicit connection. Secondly, this theorem says that it is always possible to demonstrate the inconsistency of an unsolvable system by exhibiting as a linear combination an equation that is false for all values of the unknowns. Thus both parts of the theorem

yield information through the medium of linear combinations.

The statements of Theorem 2 (b) can be strengthened to constructive assertions, that is, there is a finite computational procedure that will find the multipliers $w_1, \cdots, w_r$ in the linear combinations (and hence the solution to the transposed system) for every case. This procedure is based on the familiar technique for eliminating an unknown from the system.

THEOREM 3. *The process of elimination either yields a solution for S or exhibits an equation*

$$0x_1 + \cdots + 0x_n = d, \qquad \text{with } d \neq 0,$$

as a linear combination of the equations of S.

Although the technique of elimination is known to every algebra student in practice, it will be described here for the sake of comparison with the generalizations to follow. (Actually, "elimination" is something of a misnomer; for formal reasons, we will not eliminate an unknown, but will make all of its coefficients equal to zero.) It is applied only to systems S in which some coefficient c_{kl} is different from zero. For notational convenience, assume that $c_{11} \neq 0$, renumbering unknowns and equations if necessary. Then define a new system S' of r equations in the unknowns $x_1, \cdots, x_n$ by:

$$(S') \qquad \begin{aligned} & 0x_1 + 0x_2 + \cdots + 0x_n = 0 \\ & 0x_1 + \left(c_{22} - \frac{c_{21}c_{12}}{c_{11}}\right)x_2 + \cdots + \left(c_{2n} - \frac{c_{21}c_{1n}}{c_{11}}\right)x_n = c_2 - \frac{c_{21}}{c_{11}}c_1 \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & 0x_1 + \left(c_{r2} - \frac{c_{r1}c_{12}}{c_{11}}\right)x_2 + \cdots + \left(c_{rn} - \frac{c_{r1}c_{1n}}{c_{11}}\right)x_n = c_r - \frac{c_{r1}}{c_{11}}c_1. \end{aligned}$$

The formation of this system follows an obvious rule; it is obtained by subtracting c_{k1}/c_{11} times the first equation from the kth equation for $k = 1, \cdots, r$. Thus the coefficient of x_1 in the kth sum equation is $c_{k1} - c_{k1}c_{11}/c_{11} = 0$ for $k = 1, \cdots, r$. The seemingly redundant first equation, $0x_1 + \cdots + 0x_n = 0$, is retained to avoid the logical complications involved in considering void systems. The proof of Theorem 3 is then based on the following three assertions:

1° *Every equation of S' is a linear combination of the equations of S.*

2° *If S' is solvable, then S is solvable.*

3° *More unknowns have all zero coefficients in S' than in S.*

The details of the proof are straight-forward and will not be given. Theorem 2 (b) follows immediately from Theorem 3; however, the details of this derivation are also omitted because we shall prove a more general statement for inequalities in Section 3. At this stage, presenting the proofs might obscure the simplicity of the logical structure of the results. To emphasize this structure, the theorems

will be reproduced in a condensed form to arm the reader for the section that follows.

The core of this section is contained in the diagram:

solution → / contradiction ↓	x_1	x_2	$\cdots$	x_n	
w_1	c_{11}	c_{12}	$\cdots$	c_{1n}	$= c_1$
w_2	c_{21}	c_{22}	$\cdots$	c_{2n}	$= c_2$
$\cdot$	$\cdot$	$\cdot$		$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$		$\cdot$	$\cdot$
w_r	c_{r1}	c_{2r}	$\cdots$	c_{rn}	$= c_r$
	$= 0$	$= 0$	$\cdots$	$= 0$	$\neq 0$

It presents a schematic expression of the statement that exactly one of the two systems

$$(S)\qquad \begin{aligned} c_{11}x_1 + \cdots + c_{1n}x_n &= c_1 \\ &\cdots\cdots\cdots \\ c_{r1}x_1 + \cdots + c_{rn}x_n &= c_r \end{aligned}$$

and

$$(T)\qquad \begin{aligned} w_1c_{11} + \cdots + w_rc_{r1} &= 0 \\ &\cdots\cdots\cdots \\ w_1c_{1n} + \cdots + w_rc_{rn} &= 0 \\ w_1c_1 + \cdots + w_rc_r &\neq 0 \end{aligned}$$

is solvable. A solution to T means that S is inconsistent. Successive elimination of $x_1, \cdots, x_n$ leads either to a solution for S or to a solution for T.

The relation of solvability to consistency for linear equations should now be clear. Any confusion is caused by two different uses of the word consistency. The first (which we have called "consistency") might be called "consistency with respect to consequences" (following Church [2]) and is the notion that is used interchangeably with "solvability" because Theorem 1 is valid. The second might be called "consistency with respect to linear combinations" and leads to the following formulation of the results: Forming a linear combination is a rule of inference (*i.e.*, never leads from true statements to a false statement) for systems of linear equations. If a system S is solvable then every consequence can be derived in this manner. If S is inconsistent with respect to consequences, then it is inconsistent with respect to linear combinations. Elimination is an effective procedure for deciding whether a system is solvable or inconsistent (in either sense).

3. Systems of linear inequalities. The object of this section is to establish conditions for the solvability of general systems S of linear inequalities. We shall pattern our definitions upon those given for equations and, happily, the same theorems will be found to hold with only slight and obvious modifications.

The systems S under consideration are assumed to contain $p+q>0$ inequalities in the unknowns $x_1, \cdots, x_n$; of these $p \geqq 0$ are assumed to be strict. Thus, they can be written

$$(S) \qquad \begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &> a_i \qquad (i = 1, \cdots, p) \\ b_{j1}x_1 + \cdots + b_{jn}x_n &\geqq b_j \qquad (j = 1, \cdots, q) \end{aligned}$$

where the a_{ik}, a_i, b_{jk}, and b_j $(i=1, \cdots, p;\ j=1, \cdots, q;\ k=1, \cdots, n)$ are given real numbers that define the system S. Again, no assumption is made concerning the right hand members a_i and b_j, and the inequalities are called inhomogeneous.

An indexed set $\overline{X}$ of real numbers $(\bar{x}_1, \cdots, \bar{x}_n)$ is called a *solution* for S if all of the inequalities

$$\begin{aligned} a_{i1}\bar{x}_1 + \cdots + a_{in}\bar{x}_n &> a_i \qquad (i = 1, \cdots, p) \\ b_{j1}\bar{x}_1 + \cdots + b_{jn}\bar{x}_n &\geqq b_j \qquad (j = 1, \cdots, q) \end{aligned}$$

are true statements. The set of solutions for S is denoted by $\mathfrak{S}(S)$, and is possibly empty. An inequality,

$$d_1x_1 + \cdots + d_nx_n \mathcal{R} d,$$

where $\mathcal{R}$ is one of the relations $>$ or $\geqq$, in the unknowns $x_1, \cdots, x_n$ is called a *consequence* of S if $d_1\bar{x}_1 + \cdots + d_n\bar{x}_n \mathcal{R} d$ is a true inequality wherever $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)$ is a member of $\mathfrak{S}(S)$. If S is not solvable then *every* inequality in the unknowns $x_1, \cdots, x_n$ is a consequence of S. A system S is said to be *inconsistent* if the inequality

$$0x_1 + \cdots + 0x_n > 0$$

is a consequence of S; otherwise, it is called *consistent*. The choice of this "standard" contradiction will be explained below.

The logical connection between solvability and consistency holds without change; to emphasize the strict parallel, this theorem and those following it will be given roman numerals corresponding to their counterparts for equations.

THEOREM I. *A system S is solvable if and only if it is consistent.*

Proof. If S is solvable, choose any $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n) \in \mathfrak{S}(S)$. Then $0\bar{x}_1 + \cdots + 0\bar{x}_n$ is equal to zero by the rules for operating with real numbers and hence $0\bar{x}_1 + \cdots + 0\bar{x}_n > 0$ is not a true inequality. Hence $0x_1 + \cdots + 0x_n > 0$ is not a consequence of S and S is consistent.

On the other hand, if S is not solvable, then every inequality in $x_1, \cdots, x_n$

is a consequence of S. In particular, $0x_1 + \cdots + 0x_n > 0$ is a consequence of S and S is inconsistent.

The significant content of the theory of linear inequalities is found again in the form of the consequences that can be derived from S and the manner of their derivation, namely, as linear combinations. We must exercise some care in forming these, but no more than is indicated by the following rules for manipulating inequalities.

RULE 1. Any inequality (strict ($>$) or ordinary ($\geqq$)) still holds if it is multiplied by a *positive* number throughout.

RULE 2. Strict inequality ($>$) holds for the sum of two similarly directed inequalities if and only if strict inequality holds in at least one of the summands. Ordinary inequality ($\geqq$) can always be asserted for the sum.

RULE 3. Strict inequality ($>$) always implies ordinary inequality ($\geqq$).

These three rules make possible a precise description of the nature of the linear combinations that will be used. Rule 1 suggests that the multipliers should be restricted to be *non-negative* (a zero multiplier means that that inequality is not being used). Rule 2 says that the relation holding for the linear combination can be $>$ only if some strict inequality has a positive coefficient. Rules 2 and 3 say that the relation $\geqq$ can always be asserted for the linear combination. Following these precepts, form a multiplier scheme patterned on the scheme used for equations, with *non-negative* multipliers at the left and the sum below:

$$\begin{aligned}
u_0 \geqq 0:&\quad 0x_1 + \cdots + 0x_n > -1\\
u_1 \geqq 0:&\quad a_{11}x_1 + \cdots + a_{1n}x_n > a_1\\
&\cdots\cdots\cdots\cdots\\
u_p \geqq 0:&\quad a_{p1}x_1 + \cdots + a_{pn}x_n > a_p\\
v_1 \geqq 0:&\quad b_{11}x_1 + \cdots + b_{1n}x_n \geqq b_1\\
&\cdots\cdots\cdots\cdots\\
v_q \geqq 0:&\quad b_{q1}x_1 + \cdots + b_{qn}x_n \geqq b_q\\
\hline
&\quad d_1x_1 + \cdots + d_nx_n \mathcal{R} d
\end{aligned}.$$

The insertion of the first line is related to the choice of a standard inconsistent inequality. The coefficients of the sum are easily calculated to be:

$$\begin{aligned}
d_1 &= u_1a_{11} + \cdots + u_pa_{p1} + v_1b_{11} + \cdots + v_qb_{q1}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
d_n &= u_1a_{1n} + \cdots + u_pa_{pn} + v_1b_{1n} + \cdots + v_qb_{qn}\\
d &= -u_0 + u_1a_1 + \cdots + u_pa_p + v_1b_1 + \cdots + v_qb_q.
\end{aligned}$$

An inequality,

$$d_1x_1 + \cdots + d_nx_n \mathcal{R} d,$$

that is formed in this manner from a system S, is called a *legal linear combination* of the inequalities of S provided that $\mathfrak{R}$ is $\geqq$, *or* $\mathfrak{R}$ is $>$ and some u_i is positive ($i=0, 1, \cdots, p$). The reader should notice that this definition has been framed so as to conform to Rules 1, 2, and 3 and thus to insure the fact that every legal linear combination is a consequence. Another important fact to be noted is that every inequality in the multiplier scheme above can be proved to be a legal linear combination by choosing its multiplier equal to one and all other multipliers equal to zero. In particular, the definitions make $0x_1+\cdots+0x_n>-1$ a legal linear combination of every system.

With these definitions, the central theorem on the solvability of systems of inhomogeneous linear inequalities is:

THEOREM II. (a) *Every legal linear combination of the inequalities of S is a consequence of S.* (b) *If S is solvable then every consequence of S is a legal linear combination of the relations of S. If S is not solvable, then the inequality*

$$0x_1+\cdots+0x_n>0$$

is a legal linear combination of the relations of S.

Proof. Theorem II (a) follows immediately from the definitions and Rules 1, 2, and 3. As before, the statements of Theorem II (b) will be strengthened to constructive assertions; that is, elimination is an effective procedure for finding the multipliers $u_0, u_1, \cdots, u_p, v_1, \cdots, v_q$ in the legal linear combinations for every case. Precisely, the proof of Theorem II (b) will be based on

THEOREM III. *The process of elimination either yields a solution for S or exhibits the inequality $0x_1+\cdots+0x_n>0$ as a legal linear combination of the inequalities of S.*

Proof. The process of elimination is applied only to systems S in which some coefficient a_{il} or b_{jl} is different from zero. Assume that $l=1$, renumbering unknowns if necessary, and set about eliminating x_1 from the system. Separate the inequalities of the system into three classes (Classes I, II, III) according to whether the coefficient of x_1 is positive, negative, or zero. These classes will be distinguished notationally by the number of primes on the indices. Define a new system S' of inequalities in the unknowns $x_1, \cdots, x_n$ by:

$$0x_1+\qquad 0x_2+\cdots+\qquad 0x_n>-1$$

$$(S')\qquad 0x_1+\left(\frac{a_{i'2}}{a_{i'1}}-\frac{a_{i''2}}{a_{i''1}}\right)x_2+\cdots+\left(\frac{a_{i'n}}{a_{i'1}}-\frac{a_{i''n}}{a_{i''1}}\right)x_n>\frac{a_{i'}}{a_{i'1}}-\frac{a_{i''}}{a_{i''1}}$$

for *all* pairs i' and i'' with $a_{i'1}>0$ and $a_{i''1}<0$,

$$0x_1+\left(\frac{a_{i'2}}{a_{i'1}}-\frac{b_{j''2}}{b_{j''1}}\right)x_2+\cdots+\left(\frac{a_{i'n}}{a_{i'1}}-\frac{b_{j''n}}{b_{j''1}}\right)x_n>\frac{a_{i'}}{a_{i'1}}-\frac{b_{j''}}{b_{j''1}}$$

for *all* pairs i' and j'' with $a_{i'1}>0$ and $b_{j''1}<0$, (*S' continues on next page.*)

(*S' cont.*) $$0x_1+\left(\frac{b_{j'2}}{b_{j'1}}-\frac{a_{i''2}}{a_{i''1}}\right)x_2+\cdots+\left(\frac{b_{j'n}}{b_{j'1}}-\frac{a_{i''n}}{a_{i''1}}\right)x_n>\frac{b_{j'}}{b_{j'1}}-\frac{a_{i''}}{a_{i''1}}$$

for *all* pairs j' and i'' with $b_{j'1}>0$ and $a_{i''1}<0$,

$$0x_1+\left(\frac{b_{j'2}}{b_{j'1}}-\frac{b_{j''2}}{b_{j''1}}\right)x_2+\cdots+\left(\frac{b_{j'n}}{b_{j'1}}-\frac{b_{j''n}}{b_{j''1}}\right)x_n\geqq\frac{b_{j'}}{b_{j'1}}-\frac{b_{j''}}{b_{j''1}}$$

for *all* pairs j' and j'' with $b_{j'1}>0$ and $b_{j''1}<0$,

$$0x_1+\quad a_{i'''2}x_2+\cdots+\quad a_{i'''n}x_n>a_{i'''}$$

for *all* i''' with $a_{i'''1}=0$,

$$0x_1+\quad b_{j'''2}x_2+\cdots+\quad b_{j'''n}x_n\geqq b_{j'''}$$

for *all* j''' with $b_{j'''1}=0$.

Although it may seem formidable at first glance, the formation of the system S' follows a simple rule. Namely, the coefficient of x_1 in each inequality in Class I is made $+1$ through the use of the *positive* multipliers $1/a_{i'1}$ and $1/b_{j'1}$, while the coefficient of x_1 in each inequality of Class II is made -1 through the use of the *positive* multipliers $-1/a_{i''1}$ and $-1/b_{j''1}$. By Rule 1, the inequalities are all preserved. Then, *all* pairs of inequalities, one from Class I and one from Class II are added and the inequality type of the sum is assigned by Rule 2. The inequalities of Class III are copied, unchanged; the first inequality, $0x_1+\cdots+0x_n>-1$, is inserted in S' to avoid the logical complications involved in considering the empty system that would arise if all of the inequalities of S were in Class I or II. This characterization of the formation of S' proves:

1° *Every inequality of S' is a legal linear combination of the inequalities of S, and hence is a consequence of S.*

Another manner of viewing the formation of S' is revealed by isolating x_1 on one side of all of the inequalities of Classes I and II, then pairing the results as follows:

$$\frac{a_{i''}}{a_{i''1}}-\frac{a_{i''2}}{a_{i''1}}x_2-\cdots-\frac{a_{i''n}}{a_{i''1}}x_n>x_1>\frac{a_{i'}}{a_{i'1}}-\frac{a_{i'2}}{a_{i'1}}x_2-\cdots-\frac{a_{i'n}}{a_{i'1}}x_n$$

for *all* pairs i' and i'' with $a_{i'1}>0$ and $a_{i''1}<0$,

$$\frac{b_{j''}}{b_{j''1}}-\frac{b_{j''2}}{b_{j''1}}x_2-\cdots-\frac{b_{j''n}}{b_{j''1}}x_n\geqq x_1>\frac{a_{i'}}{a_{i'1}}-\frac{a_{i'2}}{a_{i'1}}x_2-\cdots-\frac{a_{i'n}}{a_{i'1}}x_n$$

for *all* pairs i' and j'' with $a_{i'1}>0$ and $b_{j''1}<0$,

$$\frac{a_{i''}}{a_{i''1}}-\frac{a_{i''2}}{a_{i''1}}x_2-\cdots-\frac{a_{i''n}}{a_{i''1}}x_n>x_1\geqq\frac{b_{j'}}{b_{j'1}}-\frac{b_{j'2}}{b_{j'1}}x_2-\cdots-\frac{b_{j'n}}{b_{j'1}}x_n$$

for *all* pairs j' and i'' with $b_{j'1}>0$ and $a_{i''1}<0$,

$$\frac{b_{j''}}{b_{j''1}}-\frac{b_{j''2}}{b_{j''1}}x_2-\cdots-\frac{b_{j''n}}{b_{j''1}}x_n\geqq x_1\geqq\frac{b_{j'}}{b_{j'1}}-\frac{b_{j'2}}{b_{j'1}}x_2-\cdots-\frac{b_{j'n}}{b_{j'1}}x_n$$

for *all* pairs j' and j'' with $b_{j'1}>0$ and $b_{j''1}<0$.

(Again, the reader should not ignore the possibility that either Class I or Class II might be empty, in which case there would be no such pairs; he should check that each assertion holds in this case, too.) If x_1 is dropped from each of the pairs and the result arranged with unknowns on the left and constants on the right, and if the inequalities of Class III and $0x_1+\cdots+0x_n>-1$ are adjoined to the system, the result is exactly S'.

Suppose $\bar{x}_2,\cdots,\bar{x}_n$ solve S' (note that x_1 can be given any value in this system). Then, arranging S' as in the previous paragraph, the inequalities that are not in S are:

$$(i',i''):\quad\frac{a_{i''}}{a_{i''1}}-\frac{a_{i''2}}{a_{i''1}}\bar{x}_2-\cdots-\frac{a_{i''n}}{a_{i''1}}\bar{x}_n>\frac{a_{i'}}{a_{i'1}}-\frac{a_{i'2}}{a_{i'1}}\bar{x}_2-\cdots-\frac{a_{i'n}}{a_{i'1}}\bar{x}_n$$

$$(i',j''):\quad\frac{b_{j''}}{b_{j''1}}-\frac{b_{j''2}}{b_{j''1}}\bar{x}_2-\cdots-\frac{b_{j''n}}{b_{j''1}}\bar{x}_n>\frac{a_{i'}}{a_{i'1}}-\frac{a_{i'2}}{a_{i'1}}\bar{x}_2-\cdots-\frac{a_{i'n}}{a_{i'1}}\bar{x}_n$$

$$(j',i''):\quad\frac{a_{i''}}{a_{i''1}}-\frac{a_{i''2}}{a_{i''1}}\bar{x}_2-\cdots-\frac{a_{i''n}}{a_{i''1}}\bar{x}_n>\frac{b_{j'}}{b_{j'1}}-\frac{b_{j'2}}{b_{j'1}}\bar{x}_2-\cdots-\frac{b_{j'n}}{b_{j'1}}\bar{x}_n$$

$$(j',j''):\quad\frac{b_{j''}}{b_{j''1}}-\frac{b_{j''2}}{b_{j''1}}\bar{x}_2-\cdots-\frac{b_{j''n}}{b_{j''1}}\bar{x}_n\geqq\frac{b_{j'}}{b_{j'1}}-\frac{b_{j'2}}{b_{j'1}}\bar{x}_2-\cdots-\frac{b_{j'n}}{b_{j'1}}\bar{x}_n.$$

The problem of completing $\bar{x}_2,\cdots,\bar{x}_n$ to a solution for S by a proper choice of $\bar{x}_1$ is clearly that of fitting $\bar{x}_1$ into these inequalities as required by the system immediately above them. With an eye to picking out those inequalities of S that restrict the choice of $\bar{x}_1$ most severely, define

$$\alpha'=\max_{i'}\left\{\frac{a_{i'}}{a_{i'1}}-\frac{a_{i'2}}{a_{i'1}}\bar{x}_2-\cdots-\frac{a_{i'n}}{a_{i'n}}\bar{x}_n\right\},$$

$$\beta'=\max_{j'}\left\{\frac{b_{j'}}{b_{j'1}}-\frac{b_{j'2}}{b_{j'1}}\bar{x}_2-\cdots-\frac{b_{j'n}}{b_{j'1}}\bar{x}_n\right\},$$

$$\alpha''=\min_{i''}\left\{\frac{a_{i''}}{a_{i''1}}-\frac{a_{i''2}}{a_{i''1}}\bar{x}_2-\cdots-\frac{a_{i''n}}{a_{i''1}}\bar{x}_n\right\},$$

$$\beta''=\min_{j''}\left\{\frac{b_{j''}}{b_{j''1}}-\frac{b_{j''2}}{b_{j''1}}x_2-\cdots-\frac{b_{j''n}}{b_{j''1}}x_n\right\}.$$

(We shall adopt the convention that a maximum over an empty set is $-\infty$ while a minimum over an empty set is $+\infty$. Thus, if there are no inequalities

in Class I, $\alpha' = -\infty$, *etc.*) It is important to realize these maxima and minima are actually achieved for a proper choice of the indices i', j', i'', and j'' if the sets concerned are non-empty, and so among the inequalities immediately above one can find:

$$\alpha'' > \alpha', \qquad \beta'' > \alpha', \qquad \alpha'' > \beta', \quad \text{and} \quad \beta'' \geqq \beta',$$

(where the inequalities are trivially true if the sets are empty and the convention is applied).

If $\bar{x}_1 > \alpha'$, $\bar{x}_1 \geqq \beta'$, $\bar{x}_1 < \alpha''$, *and* $\bar{x}_1 \leqq \beta''$, *then* $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ *is a solution for* S.

Proof. Since the inequalities in S of Class III are also in S', only those of Classes I and II need be checked. For these,

$$a_{i'1}\bar{x}_1 + a_{i'2}\bar{x}_2 + \cdots + a_{i'n}\bar{x}_n > a_{i'1}\alpha' + a_{i'2}\bar{x}_2 + \cdots + a_{i'n}\bar{x}_n$$
$$\geqq (a_{i'} - a_{i'2}\bar{x}_2 - \cdots - a_{i'n}\bar{x}_n) + (a_{i'2}\bar{x}_2 + \cdots + a_{i'n}\bar{x}_n) = a_{i'}$$

for all i' with $a_{i'1} > 0$,

$$a_{i''1}\bar{x}_1 + a_{i''2}\bar{x}_2 + \cdots + a_{i''n}\bar{x}_n > a_{i''1}\alpha'' + a_{i''2}\bar{x}_2 + \cdots + a_{i''n}\bar{x}_n$$
$$\geqq (a_{i''} - a_{i''2}\bar{x}_2 - \cdots - a_{i''n}\bar{x}_n) + (a_{i''2}\bar{x}_2 + \cdots + a_{i''n}\bar{x}_n) = a_{i''}$$

for all i'' with $a_{i''1} < 0$,

$$b_{j'1}\bar{x}_1 + b_{j'2}\bar{x}_2 + \cdots + b_{j'n}\bar{x}_n > b_{j'1}\beta' + b_{j'2}\bar{x}_2 + \cdots + b_{j'n}\bar{x}_n$$
$$\geqq (b_{j'} - b_{j'2}\bar{x}_2 - \cdots - b_{j'n}\bar{x}_n) + (b_{j'2}\bar{x}_2 + \cdots + b_{j'n}\bar{x}_n) = b_{j'}$$

for all j' with $b_{j'1} > 0$,

$$b_{j''1}\bar{x}_1 + b_{j''2}\bar{x}_2 + \cdots + b_{j''n}\bar{x}_n \geqq b_{j''1}\beta'' + b_{j''2}\bar{x}_2 + \cdots + b_{j''n}\bar{x}_n$$
$$\geqq (b_{j''} - b_{j''2}\bar{x}_2 - \cdots - b_{j''n}\bar{x}_n) + (b_{j''2}\bar{x}_2 + \cdots + b_{j''n}\bar{x}_n) = b_{j''}$$

for all j'' with $b_{j''1} < 0$. This proves that S is solved by $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$.

To choose an $\bar{x}_1$ satisfying these conditions, consider the four exhaustive cases, corresponding to the four possible orderings of α' with β' and α'' with β'', separately:

CASE 1. $\beta' \leqq \alpha'$ and $\alpha'' \leqq \beta''$.
Choose $\bar{x}_1$ so that $\alpha' < \bar{x}_1 < \alpha''$.

CASE 2. $\beta' \leqq \alpha'$ and $\beta'' < \alpha''$.
Choose $\bar{x}_1$ so that $\alpha' < \bar{x}_1 < \beta''$.

CASE 3. $\alpha' < \beta'$ and $\alpha'' \leqq \beta''$.
Choose $\bar{x}_1$ so that $\beta' < \bar{x}_1 < \alpha''$.

CASE 4. $\alpha' < \beta'$ and $\beta'' < \alpha''$.
Choose $\bar{x}_1$ so that $\beta' \leqq \bar{x}_1 \leqq \beta''$.

This proves the crucial fact:

2° *If* S' *is solvable, then* S *is solvable.*

Suppose that the process of elimination is applied several, say h, times to yield successively the systems S, S', S'', $\cdots$, $S^{(h)}$. It is clear that 1° and 2° still hold, with $S^{(h)}$ replacing S'. The only detail that needs verifying, namely, that legal linear combinations of legal linear combinations are legal linear combinations, is obvious and tedious to write out. The termination of the process is insured by a third property of S' relative to S.

3° *More unknowns have all zero coefficients in S' than in S.*

This is clear, since any unknown with all zero coefficients in S still has all zero coefficients in S' while x_1 had a non-zero coefficient in S (some a_{i1} or b_{j1} was assumed non-zero) and has all zero coefficients in S'.

Property 3° insures that the process of elimination must end after a finite number of steps (no more than n) with a system $S^{(h)}$ in which all of the unknowns have all zero coefficients. If all the right hand members of the strict (or ordinary) inequalities are negative (or non-positive) then any set of numbers $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n)$ solves $S^{(h)}$ and hence S is solvable by h applications of 2°. Otherwise, $0x_1 + \cdots + 0x_n > d$ with some $d \geqq 0$, or $0x_1 + \cdots + 0x_n \geqq d$ with some $d > 0$ appears in $S^{(h)}$. In this case, the proof is completed by writing down the two multiplier schemes:

$$\begin{array}{ll} 1\text{:} & 0x_1 + \cdots + 0x_n > d, \quad \text{with any } d \geqq 0 \\ d\text{:} & 0x_1 + \cdots + 0x_n > -1 \\ \hline & 0x_1 + \cdots + 0x_n > 0 \end{array}$$

$$\begin{array}{ll} 1\text{:} & 0x_1 + \cdots + 0x_n \geqq d, \quad \text{with any } d > 0 \\ d\text{:} & 0x_1 + \cdots + 0x_n > -1 \\ \hline & 0x_1 + \cdots + 0x_n > 0 \end{array}$$

(The reader should verify that the result is a legal linear combination in each case.) Hence, $0x_1 + \cdots + 0x_n > 0$ is a legal linear combination of $S^{(h)}$ and can be exhibited as a legal linear combination of the inequalities of S (and possibly $0x_1 + \cdots + 0x_n > -1$) by at most $h+1$ applications of 1°. This completes the proof of Theorem III.

Before returning to the proof of Theorem II (b), these results will be summarized in a multiplier scheme. This diagram expresses the fact that *exactly* one of the two systems listed there admits a solution. A solution to T means that S is inconsistent. Successive elimination of $x_1, \cdots, x_n$ leads either to a solution for S or to a solution for T.

solution → / contradiction		x_1	x_2	$\cdots$	x_n	
	$u_0 \geqq 0$	0	0	$\cdots$	0	> -1
not	$u_1 \geqq 0$	a_{11}	a_{12}	$\cdots$	a_{1n}	$> a_1$
all	$u_2 \geqq 0$	a_{21}	a_{22}	$\cdots$	a_{2n}	$> a_2$
zero	⋮	⋮	⋮		⋮	⋮
	$u_p \geqq 0$	a_{p1}	a_{p2}	$\cdots$	a_{pn}	$> a_p$
	$v_1 \geqq 0$	b_{11}	b_{12}	$\cdots$	b_{1n}	$\geqq b_1$
	$v_2 \geqq 0$	b_{21}	b_{22}	$\cdots$	b_{2n}	$\geqq b_2$
	⋮	⋮	⋮		⋮	⋮
	$v_q \geqq 0$	b_{q1}	b_{q2}	$\cdots$	b_{qn}	$\geqq b_q$
		$= 0$	$= 0$	$\cdots$	$= 0$	$= 0$

The system S is defined by

$$a_{i1}x_1 + \cdots + a_{in}x_n > a_i \qquad (i = 1, \cdots, p)$$
$$b_{j1}x_1 + \cdots + b_{jn}x_n \geqq b_j \qquad (j = 1, \cdots, q).$$

The system T is defined by

$$u_1a_{1l} + \cdots + u_pa_{pl} + v_1b_{1l} + \cdots + v_qb_{ql} = 0 \qquad (l = 1, \cdots, n)$$
$$-u_0 + u_1a_2 + \cdots + u_pa_p + v_1b_1 + \cdots + v_qb_q = 0$$

with all of the unknowns $u_0, u_1, \cdots, u_p, v_1, \cdots, v_q$ non-negative and not all of the $u_0, u_1, \cdots, u_p$ equal to zero.

Proof of Theorem II(b). The case for systems without solutions is stated directly in Theorem III and so it is only necessary to show that, for solvable systems, every consequence is a legal linear combination. Assume that the strict inequality $d_1x_1 + \cdots + d_nx_n > d$ is a consequence of S. Thus, if $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n)$ is any solution for S, then $d_1\bar{x}_1 + \cdots + d_n\bar{x}_n$ is a number that is *larger* than d. This means the *insolvability* of the system U (to save rewriting the system, multipliers have been included):

$$(U) \qquad \begin{aligned} u_0 \geqq 0: &\quad 0x_1 + \cdots + 0x_n > -1 \\ u_i \geqq 0: &\quad a_{i1}x_1 + \cdots + a_{in}x_n > a_i \qquad (i = 1, \cdots, p) \\ v_j \geqq 0: &\quad b_{j1}x_1 + \cdots + b_{jn}x_n \geqq b_j \qquad (j = 1, \cdots, q) \\ t \geqq 0: &\quad -d_1x_1 - \cdots - d_nx_n \geqq -d. \end{aligned}$$

By Theorem III, there exist non-negative multipliers $\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_p, \bar{v}_1, \cdots, \bar{v}_q, \bar{t}$ with not all of $\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_p$ equal to zero, such that

$$\bar{x}_l: \quad \bar{u}_1 a_{1l} + \cdots + \bar{u}_p a_{pl} + \bar{v}_1 b_{1l} + \cdots + \bar{v}_q b_{ql} - \bar{l}d = 0 \quad (l = 1, \cdots, n)$$

$$-1: \quad -\bar{u}_0 + \bar{u}_1 a_1 + \cdots + \bar{u}_p a_p + \bar{v}_1 b_1 + \cdots + \bar{v}_q b_q - \bar{l}d = 0.$$

If $\bar{l}$ were known to be positive, then all of these equations could be divided by it, establishing $d_1x_1 + \cdots + d_nx_n > d$ as a legal linear combination of the inequalities of S. Recall that S is assumed to be solvable and let $\bar{x}_1, \cdots, \bar{x}_n$ be a solution. These have been written as multipliers at the left of the last system, using -1 to multiply the last equation. Multiplying and adding,

$$\bar{u}_0 + \sum_i \bar{u}_i(a_{i1}\bar{x}_1 + \cdots + a_{in}\bar{x}_n - a_i) + \sum_j \bar{v}_j(b_{j1}\bar{x}_1 + \cdots + b_{jn}\bar{x}_n - b_j)$$

$$= \bar{l}(d_1\bar{x}_1 + \cdots + d_n\bar{x}_n - d).$$

Since the left side of this equation is positive, the right side is also, and hence $\bar{l} > 0$. Hence $d_1x_1 + \cdots + d_nx_n > d$ is a legal linear combination of the inequalities of S with multipliers $\bar{u}_0/\bar{l}, \bar{u}_1/\bar{l}, \cdots, \bar{u}_p/\bar{l}, \bar{v}_1/\bar{l}, \cdots, \bar{v}_q/\bar{l}$. If the consequence is an ordinary inequality $d_1x_1 + \cdots + d_nx_n \geqq d$, a few changes must be made in the proof. The inequality $-d_1x_1 - \cdots - d_nx_n \geqq -d$ in U should be replaced by $-d_1x_1 - \cdots - d_nx_n > -d$. Using the same notation for multipliers, the same transposed system of equations is obtained, with not all of $\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_p$, *and* $\bar{l}$ equal to zero. If $\bar{l}$ is assumed zero, we are in the previous case (with not all of $\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_p$ equal to zero) and can conclude $\bar{l} > 0$ as before, a contradiction. Hence $\bar{l}$ is positive and $d_1x_1 + \cdots + d_nx_n \geqq d$ is a legal linear combination of the inequalities of S with multipliers $\bar{u}_0/\bar{l}, \bar{u}_1/\bar{l}, \cdots, \bar{u}_p/\bar{l}, \bar{v}_1/\bar{l}, \cdots, \bar{v}_q/\bar{l}$. This completes the proof of Theorem II (b).

4. An infinite example. Of course, the results of Section 3 can be paraphrased in logical terms exactly in the form of the conclusion to Section 2. At this point the reader may be led to the erroneous belief that consistency with respect to consequences is always equivalent to consistency with respect to linear combinations for linear systems. The following (infinite) system shows that this is not so.

$$(S) \qquad x > -1/n \qquad (n = 1, 2, 3, \cdots).$$

Clearly $x \geqq 0$ is a consequence of this system, yet is not a linear combination of any number of inequalities of S. We infer that

$$(S_1) \qquad -x > 0, \quad x > -1/n \qquad (n = 1, 2, 3, \cdots)$$

is inconsistent with respect to consequences but is consistent with respect to linear combinations. The natural way out of this dilemma is to add new rules of inference based on limiting operations.

5. Acknowledgments and historical remarks. The author is indebted to T. S. Motzkin, who first suggested that his "transposition theorem" [6] might be viewed as asserting the disjoint alternatives of solvability or contradiction *via* linear combination. He also called attention to a remark of Fourier [5] that elimination was a natural method for solving linear inequalities. However, the

author takes full responsibility for the logical consequences of these ideas.

Elimination, as a method for solving linear inequalities, has been used for theoretical purposes by a number of authors, notably by Dines [4]. However, his treatment seems to conceal rather than emphasize the parallel with equations.

The distinction between logical consequences and provable consequences for logistic systems was first made by Tarski [7]. Our definition of the logical consequences of a system of linear equations or inequalities is in an obvious way parallel to Tarski's definition of logical consequence, but the two are not the same. They might be compared in the following manner. The logical consequences, in the sense of this paper, of a system of equations or inequalities are the same as the logical consequences, in Tarski's sense, of the system together with some categorical system of postulates for the real numbers—provided that the unknowns in the equations or inequalities, before considering the consequences in Tarski's sense, are first replaced by new symbols, which play the role of primitive constants, and which do not appear elsewhere. The author is indebted to A. Church for unraveling the relation between the two ideas; the general metatheorem parallel to Theorems 1 and I, and valid for functional calculi of all orders is stated by Church in [2].

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SELECTED SELECTION THEOREMS

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1. Introduction. The theory of selections, a systematic account of which appears in the *Annals of Mathematics* [6] [7] [8], contains two theorems which are both of general interest and relatively easy to prove ([6; part of Theorem 3.2″] and [7; part of Theorem 1.2]). The purpose of this note is to make these theorems, together with their proofs and some applications, accessible to the casual reader, who might find the sheer bulk of the more detailed *Annals* investigation somewhat discouraging.

If X and Y are topological spaces, and ϕ a function from X to the subsets of Y, then a *selection* for ϕ is a continuous $f: X \to Y$ such that $f(x) \in \phi(x)$ for every x in X. All other unfamiliar concepts appearing in the following theorems will be defined at the end of this introduction.

THEOREM 1 [6]. *If X is a paracompact space, then every lower semi-continuous function ϕ from X to the non-empty, closed, convex subsets of a Banach space Y admits a selection.*

THEOREM 2 [7]. *If X is paracompact and zero-dimensional, and if Y is a complete metric space, then every lower semi-continuous function ϕ from X to the non-empty, closed subsets of Y admits a selection.*

The difference between Theorem 1 and Theorem 2, it should be noted, is that Theorem 1 restricts the *range* of ϕ, while Theorem 2 restricts its *domain*. Intermediate results, where $\dim X < n$ and where the range of ϕ is subjected to restrictions weaker than those in Theorem 1, can be found in [7; Theorem 1.2].

It is shown in [6; Theorem 3.2″] that the converse of Theorem 1 is also true; in other words, any T_1-space X which has the property stated in Theorem 1 is paracompact. This provides a new characterization of paracompactness.

We now state the three most important corollaries to Theorems 1 and 2. The first of them was first proved by R. Bartle and L. Graves [2]; the second and third are Corollaries 1.4 and 1.5 of [7].

COROLLARY 1 (Bartle-Graves). *If X and Y are Banach spaces, and if $u: Y \to X$ is continuous, linear, and onto, then there exists a continuous $f: X \to Y$ such that $uf(x) = x$ for every $x \in X$.***

COROLLARY 2. *If X is a zero-dimensional paracompact space, Y a complete metric space, and if $u: Y \to X$ is continuous, open, and onto, then there exists a continuous $f: X \to Y$ such that $uf(x) = x$ for every $x \in X$.*

* The results in this paper were obtained while the author was an A. E. C. fellow.

** One can *not* always pick f to be *linear*, since this would imply the existence of a continuous projection from Y onto the null-space of u, which need not exist [9]. One can, however, at least pick f such that $f(\alpha x) = \alpha f(x)$ for every scalar α [6; Proposition 7.2].

COROLLARY 3. *Let X be a zero-dimensional paracompact space, let C be the complex numbers, and let $g: X \to C$ be continuous. Also let w be any polynomial with complex coefficients.*† *Then there exists a continuous $f: X \to C$ such that $w(f(x)) = g(x)$ for every x in X.*‡

We conclude this introduction with a quick review of our principal concepts. Let X be a topological space. A function ϕ from X to the subsets of a topological space Y is *lower semicontinuous* if $\{x \in X \mid \phi(x) \cap U \neq \phi\}$ is open in X for every open $U \subset Y$. If $\mathcal{U}$ and $\mathcal{W}$ are coverings of X, then $\mathcal{U}$ is a *refinement* of $\mathcal{W}$ if every $V \in \mathcal{U}$ is a subset of some $W \in \mathcal{W}$. A collection $\mathcal{U}$ of subsets of X is *locally finite* if every $x \in X$ has a neighborhood which intersects only finitely many $V \in \mathcal{U}$. We call X *paracompact* if it is Hausdorff, and if every open covering of X has an open, locally finite refinement. (It is known [3] [10] that every metric space and every compact Hausdorff space is paracompact, and that every paracompact space is normal; for more details, see [4; p. 156].) Finally, X is *zero-dimensional* (dim $X = 0$) if every finite open covering of X has a disjoint, finite, open refinement. This implies (and for compact Hausdorff or separable metric spaces is implied by) the condition that X has a base consisting of sets which are both open and closed.

2. Proof of Theorem 1. Before embarking on the proof of Theorem 1, we state, and for completeness prove, a useful and standard result about "partitions of unity" on paracompact spaces [5].

PROPOSITION 1. *If $\{V_\alpha\}_{\alpha \in A}$ is a locally finite, open covering of normal space X, then there exists a family $\{p_\alpha\}_{\alpha \in A}$ of continuous functions from X to the closed unit interval with the following properties*:

(a) *p_α vanishes outside V_α,*

(b) *$\Sigma_{\alpha \in A} p_\alpha(x) = 1$ for every x in X.*

Proof. By a theorem of Dieudonné, [3; Theorem 6], there exists an open covering $\{W_\alpha\}_{\alpha \in A}$ of X such that $\overline{W}_\alpha \subset V_\alpha$ for every $x \in X$. By the normality of X, we can find continuous functions $q_\alpha: X \to [0, 1]$ such that

$$q_\alpha(x) = 1 \qquad x \in \overline{W}_\alpha,$$
$$q_\alpha(x) = 0 \qquad x \in X - V_\alpha.$$

Now define $p_\alpha: X \to [0, 1]$ by

$$p_\alpha(x) = \frac{q_\alpha(x)}{\sum_{\beta \in A} q_\beta(x)}.$$

† As the proof shows, w may actually be any open (in particular, any analytic) map from a subset of C into C whose range contains $g(X)$.

‡ This corollary can be used to prove the known result that every normal operator A on a Hilbert space has an n-th root lying in the smallest strongly closed *-algebra of operators containing A.

The family $\{p_\alpha\}_{\alpha\in A}$ satisfies all our requirements.

Turning now to the proof of Theorem 1, we begin with the following lemma, which will be used in an inductive proof of the theorem.

LEMMA 1. *If X is paracompact, Y a normed linear space, ψ a lower semi-continuous function from X to the non-empty convex subsets of Y, and $r>0$, then there exists a continuous $f: X\to Y$ such that $f(x)\in S_r(\psi(x))$ for every $x\in X$.**

Proof. For every $y\in Y$, let $U_y=\{x\in X \mid y\in S_r(\psi(x))\}$. Then

$$U_y = \{x \in X \mid \psi(x) \cap S_r(y) \neq \varnothing\},$$

and it follows from the definition of lower semi-continuity that every U_y is open in X. Hence $\{U_y\}_{y\in Y}$ is an open covering of X, and therefore has an open locally finite refinement $\{V_\alpha\}_{\alpha\in A}$.† Now pick the family $\{p_\alpha\}_{\alpha\in A}$ of continuous functions from X to the interval $[0, 1]$ as in Proposition 1. Also pick, for each $\alpha\in A$, a $y(\alpha)\in Y$ such that $V_\alpha\subset U_{y(\alpha)}$. The desired $f: X\to Y$ can now be defined by

$$f(x) = \sum_{\alpha\in A} p_\alpha(x)y(\alpha).$$

To see that this works, observe first that each $x\in X$ has a neighborhood U intersecting only finitely many V_α, and on this U, f is the sum of finitely many continuous functions. Hence every $x\in X$ has a neighborhood on which f is continuous, and therefore f is continuous on X. Moreover, for each $x\in X$, $f(x)$ is a *convex*, linear combination of finitely many $y(\alpha)$, all of which lie in the *convex* set $S_r(\psi(x))$, and hence $f(x)\in S_r(\psi(x))$. This completes the proof of the lemma.

Proof of Theorem 1 itself. We are going to construct a sequence of continuous functions $f_i: X\to Y$ such that, for every $x\in X$,

$$\text{(a)}\quad f_i(x) \in S_{2^{-i+2}}(f_{i-1}(x)) \qquad (i = 2, 3, \cdots)$$

$$\text{(b)}\quad f_i(x) \in S_{2^{-i}}(\phi(x)) \qquad (i = 1, 2, \cdots).$$

This will be sufficient, because then (by (a)) the sequence $\{f_i\}_{i=1}^{\infty}$ is uniformly Cauchy, and therefore converges uniformly to a continuous $f: X\to Y$, and it follows from (b) that $f(x)\in\phi(x)$ for every x in X.

We construct $\{f_i\}_{i=1}^{n}$ by induction. The existence of a function f_1, satisfying (b) for $i=1$, follows immediately from Lemma 1. Suppose that $f_1, \cdots, f_n$ have been constructed to satisfy (a) and (b) for $i=1, \cdots, n$, and let us construct f_{n+1} to satisfy (a) and (b) for $i=n+1$.

For each $x\in X$, let

$$\phi_{n+1}(x) = \phi(x) \cap (S_{2^{-n-1}}(f_n(x)));$$

* If B is a subset of a metric space Y with metric ρ, and if $r>0$, then $S_r(B)$ denotes $\{y\in Y \mid \rho(y, B)<r\}$.

† We assume that $V_\alpha\neq V_\beta$ for $\alpha\neq\beta$.

then $\phi_{n+1}(x)$ is never empty by our induction hypothesis. Let us check that ϕ_{n+1} is also lower semi-continuous. We must show that, if U is open in Y, then $V=\{x\in X \mid \phi_{n+1}(x)\cap U\neq\varnothing\}$ is open in X; to do this, we will show that every $x_0\in V$ has a neighborhood contained in V. Now for $x_0\in V$, pick $y_0\in\phi(x_0)$, and a positive $\lambda<2^{-n}$ such that $y_0\in S_\lambda(f_n(x_0))$. Let

$$W_1=\{x\in X \mid \phi(x)\cap(S_\lambda(f_n(x_0))\cap U)\neq\varnothing\};$$
$$W_2=\{x\in X \mid f_n(x)\in S_{2^{-n}-\lambda}(f_n(x_0))\}.$$

Then W_1 is open because ϕ is lower semi-continuous, W_2 is open because f_n is continuous, and $x_0\in(W_1\cap W_2)\subset V$.

Since ϕ_{n+1} is lower semi-continuous, we can apply Lemma 1 to find a continuous $f_{n+1}:X\to Y$ such that

$$f_{n+1}(x)\in S_{2^{-n-1}}(\phi_{n+1}(x))$$

for every $x\in X$. But then

$$f_{n+1}(x)\in S_{2^{-n+1}}(f_n(x)),$$

which is (a), and

$$f_{n+1}(x)\in S_{2^{-n-1}}(\phi(x)),$$

which is (b).

3. Proof of Theorem 2. In the proof of Theorem 2, we shall need the equivalence of (a) and (b) in the following elementary and known result, which we prove here for completeness. Note that for *compact* X the equivalence of (a) and (b) is obvious.

PROPOSITION 2. *The following three properties of a normal space X are equivalent*:

(a) *Every finite open covering of X has a disjoint finite open refinement* (*i.e.*, dim $X=0$).

(b) *Every locally finite open covering of X has a disjoint open refinement.*

(c) *If $A\subset V\subset X$, with A closed and V open, then there exists an open and closed $W\subset X$ with $A\subset W\subset V$.*

Proof (a)→(c): Let A and V be as in (c). Then $\{V, X-A\}$ is a finite open covering of X, which by assumption has a finite, disjoint open refinement $\mathcal{U}$. Let W be the union of all elements of $\mathcal{U}$ which are subsets of V. The W satisfies all our requirements.

(c)→(b): Let $\{U_\alpha\}_{\alpha\in A}$ be a locally finite open covering of X. Again applying Dieudonné's theorem [3; Theorem 6], we can find an open covering $\{V_\alpha\}$ of X such that $\overline{V}_\alpha\subset U_\alpha$ for all α. By assumption we can now find, for each α, an open and closed W_α such that $\overline{V}_\alpha\subset W_\alpha\subset U_\alpha$. Now well-order the index set A, and for each $\alpha\in$A, let $R_\alpha=W_\alpha-\bigcup_{\beta<\alpha}W_\beta$. Now for each α, $\bigcup_{\beta<\alpha}W_\alpha$ is the union of a locally finite collection of closed sets, and is therefore closed. Hence each R_α is

open, and $\{R_\alpha\}_{\alpha\in A}$ is a disjoint, open refinement of $\{U_\alpha\}_{\alpha\in A}$.

(b)→(a): This is obvious.

Having established Proposition 2, we now turn to the proof of Theorem 2. It begins with the following lemma, which is a precise analogue of Lemma 1.

LEMMA 2. *If X is paracompact and zero-dimensional, Y a metric space, ψ a lower semi-continuous function from X to the non-empty subsets of Y, and if $r>0$, then there exists a continuous $f:X\to Y$ such that $f(x)\in S_r(\psi(x))$ for every x in X.*

Proof. For every $y\in Y$, let $U_y=\{x\in X \mid y\in S_r(\psi(x))\}$. Just as in the proof of Lemma 1, each U_y is open in X, and $\{U_y\}_{y\in Y}$ is an open covering of X. Since X is paracompact, $\{U_y\}_{y\in Y}$ has an open, locally finite refinement $\mathcal{R}$. Since X is a zero-dimensional normal space, $\mathcal{R}$ has an open *disjoint* refinement $\mathcal{W}$, by Proposition 2. For each $W\in\mathcal{W}$, pick a $y(W)\in Y$ such that $W\subset U_{y(W)}$, and define $f:X\to Y$ by

$$f(x) = y(W) \qquad \text{if } x\in W.$$

It is obvious that f satisfies all our requirements. This completes the proof of the lemma.

Theorem 2 now follows from Lemma 2 *in verbatim* the same way that Theorem 1 followed from Lemma 1.

4. Proof of the corollaries. To see how Corollary 1 follows from Theorem 1, we begin by observing that, by the open mapping theorem of Banach [1; p. 38, (1)], u is open (*i.e.*, maps open sets into open sets). Define ϕ from X to the non-empty, closed, convex subsets of Y by $\phi(x)=u^{-1}(x)$. The openness of u immediately implies (in fact, is equivalent to) the lower semi-continuity of ϕ. Since X is a Banach space, it is metric, and therefore paracompact. Hence, by Theorem 1, there exists a selection f for ϕ, and this f obviously satisfies our requirements.

The proof that Theorem 2 implies Corollary 2 proceeds in *precisely* the same fashion, except that this time it is *assumed* that u is open.

Finally, observe how Corollary 3 follows from Theorem 2. The polynomial w, like any analytic function, maps open sets into open sets (see, for instance, [11]). This implies that the function ϕ from X to the closed, non-empty subset of C, defined by $\phi(x)=w^{-1}(g(x))$, is lower semi-continuous. Hence, by Theorem 2, there exists a selection f for ϕ, and this f clearly does the trick.

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ESSENTIAL MAPPINGS*

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1. Introduction. The reader is requested to perform (at least mentally) the following experiment. Tie the ends of a piece of string together to form a loop, drive two pegs into a plane surface, and place the loop around the pegs as in Figure 1.

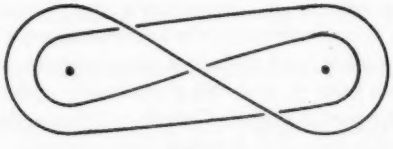

FIG. 1

Now, if one attempts to pull the string from around the pegs, keeping it on the plane and not lifting it over the pegs, it will be discovered that this is impossible. However, if either of the pegs is removed, then it is easy to pull the loop free from the other peg.

In this paper we discuss a generalization of this phenomenon. It is shown that if n is any positive integer, and n pegs are inserted into the plane surface, then it is possible to place the loop around the pegs in such a way that it is impossible to slide the loop free from the pegs, but if any one of the pegs is removed then it is possible to slide the loop free from the remaining $n-1$ pegs. Such a position of the loop for the case $n=3$ is shown in Figure 2.

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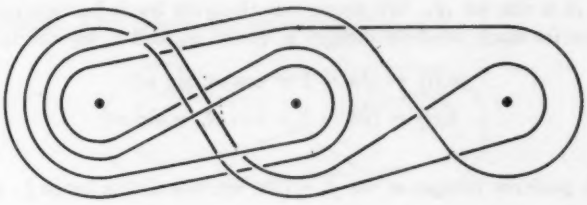

FIG. 2

2. A theorem about essential mappings. Let E^2 be Euclidean 2-space, and let I be the closed interval of real numbers from 0 to 1. If X is a subset of E^2, a *path* in X is a mapping (continuous function) on I into X. A path f is a *loop* if $f(0)=f(1)$. If $p=f(0)=f(1)$, then f is a *loop at* p.

If f and g are paths in E^2 and $f(1)=g(0)$, then fg is the path h for which

$$h(t) = f(2t), \qquad 0 \leqq t \leqq \tfrac{1}{2},$$
$$h(t) = g(2t-1), \qquad \tfrac{1}{2} \leqq t \leqq 1.$$

If f, g and h are paths for which $f(1)=g(0)$ and $g(1)=h(0)$, then fgh is defined to be the path $(fg)h$. Likewise, $fghk$ is $(fgh)k$, *etc.*

If f is a path, then f^{-1} is the path defined by $f^{-1}(t)=f(1-t)$.

If f and g are paths in X, $f(0)=g(0)$ and $f(1)=g(1)$, then f and g are *equivalent in* X if and only if there exists a continuous function H on $I\times I$ into X such that:

(i) $H(0, t)=f(t)$ for $0\leqq t\leqq 1$;
(ii) $H(1, t)=g(t)$ for $0\leqq t\leqq 1$;
(iii) $H(s, 0)=f(0)=g(0)$ for $0\leqq s\leqq 1$;
(iv) $H(s, 1)=f(1)=g(1)$ for $0\leqq s\leqq 1$.

The basic facts concerning paths, loops and equivalence of paths are contained in [1].

The *constant loop* at p is the function f for which $f(t)=p$ for each t in I. A loop g in a set X is *essential* in X if and only if g is not equivalent in X to the constant loop at $g(0)$.

We now state and prove a theorem which is suggested by the string experiment.

THEOREM 1. *If A is a non empty finite subset of E^2, then there exists a loop g which is essential in E^2-A, but which is not essential in E^2-B for every proper subset B of A.*

Proof. For each positive integer n, let a_n be the point $(2n-1, 0)$, and let A_n be the set whose members are $a_1, \cdots, a_n$. If the set A of our theorem has n members, then there exists a homeomorphism of E^2 onto itself which maps A onto the set A_n. It follows that we may assume without loss of generality that

for some n, A is the set A_n. We prove our theorem by induction on n.

We define for each positive integer n, paths u_n and v_n by the formulas

$$u_n(t) = (2n - 1 + \cos \pi t, \sin \pi t)$$
$$v_n(t) = (2n - 1 - \cos \pi t, -\sin \pi t)$$

for $0 \leqq t \leqq 1$.

For each positive integer n, let $p_n = (2n, 0)$. We define loops f_n at p_n inductively by letting

$$f_1 = u_1 v_1,$$
$$f_{n+1} = u_{n+1} f_n u_{n+1}^{-1} v_{n+1}^{-1} f_n^{-1} v_{n+1}.$$

We let R_n be the range of f_n. It is obvious that there exists a retract of $E^2 - A_n$ onto R_n, and consequently f_n is essential in $E^2 - A_n$ if and only if f_n is essential in R_n. We wish to prove by induction that f_n is essential in R_n for each positive integer n. It is well known that f_1 is essential in R_1. Let us assume that f_k is essential in R_k, and show that this implies that f_{k+1} is essential in R_{k+1}. We define a loop g at p_k by letting $g = u^{-1}{}_{k+1} f_{k+1} u_{k+1}$. It is well known that f_{k+1} is essential in R_{k+1} if and only if g is essential in R_{k+1}. However, g is equivalent in R_{k+1} to $h = f_k u_{k+1}^{-1} v_{k+1}^{-1} f_k^{-1} v_{k+1} u_{k+1}$. If we can prove that h is essential in R_{k+1}, then it will follow that g and f_{k+1} are also essential in R_{k+1}.

Let C be the circle with center a_{k+1} and radius 1. Both R_k and C are homeomorphic to connected complexes, $R_{k+1} = R_k \cup C$, and $R_k \cap C$ consists of the single point p_k. It follows from a well known theorem (see [1] p. 179) that the fundamental group of R_{k+1} is the free product of the fundamental groups of R_k and C. Since f_k, $u_{k+1}^{-1} v_{k+1}^{-1}$, f_k^{-1}, and $v_{k+1} u_{k+1}$ are essential loops in R_k, C, R_k and C respectively, it follows that h is an essential loop in R_{k+1}. We have thus proved that f_n is essential in $E^2 - A_n$ for each n.

Now let B be a proper subset of A_n. We wish to prove that f_n is not essential in $E^2 - B$. In order to accomplish this, we prove by induction the following slightly more general proposition:

If B is a set which consists of a finite number of the points $a_1, a_2, a_3, \cdots$, and $A_n - B$ is non-empty, then f_n is not essential in $E^2 - B$.

The above proposition is obvious for $n = 1$. Let us assume that it is true for $n = k$ and that B is a finite subset of $a_1, a_2, a_3, \cdots$ such that $A_{k+1} - B$ is non-empty. If $A_k - B$ is non-empty, then f_k is non-essential in $E^2 - B$, and hence f_{k+1} is equivalent to $u_{k+1} u_{k+1}^{-1} v_{k+1}^{-1} v_{k+1}$ which is equivalent to a constant loop. If $A_k - B$ is empty, then $a_{k+1} \notin B$, and it follows that $v_{k+1} u_{k+1}$ and $u_{k+1}^{-1} v_{k+1}^{-1}$ are not essential in $E^2 - B$; and hence f_{k+1} is not essential in $E^2 - B$.

3. Conclusion. Although there is an obvious correspondence between the physical problem concerning the string which we stated in section 1 and the theorem which we proved in section 2, there is not a complete analogy between

the two problems. This is seen by examining Figure 3.

FIG. 3

The strings pictured in Figures 1 and 3 project onto the same loop in the plane, and this loop is not essential in the complement of either of the two points. However, if one of the pegs is removed, the string in Figure 1 can be slid free from the remaining peg, while the string in Fig. 3 cannot be slid free from the remaining peg. We now formulate a mathematical theorem in 3-space which more nearly approximates the string problem.

Let E^3 be Euclidean 3-space. If X is a subset of E^3, a *simple loop* in X is a mapping f of I into X which has the property that $f(t)=f(s)$ if and only if either $t=s$ or one of the numbers t, s is 0 and the other is 1. A simple loop f in X is *isotopically non-essential* in X if and only if there exists a mapping H of $I\times I$ into X such that:

(i) $H(0, t)=f(t)$ for $0\leqq t\leqq 1$;

(ii) $H(s, t)=f(0)$ if $t=0$, $t=1$, or $s=1$;

(iii) for each fixed s, $0\leqq s<1$, the function F_s defined by $F_s(t)=H(s, t)$ is a simple loop.

A simple loop in X that is not isotopically non-essential in X is said to be *isotopically essential* in X.

The following theorem is a mathematical formulation of the string problem.

THEOREM 2. *If A is the union of a finite number of parallel lines in E^3, then there exists a simple loop in E^3-A which is isotopically essential in E^3-A, but which is isotopically non-essential in E^3-B for every proper subset B of A.*

Proof. Let n be the number of lines in A. We may without loss of generality assume that A is the set of all points $(2k-1, 0, r)$ for which r is real, k is an integer, and $1\leqq k\leqq n$. Let f_n be the plane loop which was defined in the proof of Theorem 1, let $X(t)$ be the first coordinate of $f_n(t)$ and let $Y(t)$ be the second coordinate of $f_n(t)$. We define paths g and h in E^3-A by letting $g(t)=(X(t), Y(t), 2t)$ and $h(t)=(2n+\sin \pi t, 0, 1+\cos \pi t)$ for $0\leqq t\leqq 1$. Since $g(1)=h(0)$, the product path gh is defined. It is easily verified that gh is a simple loop in E^3-A, and that gh has the properties of the loop whose existence is asserted in our theorem.

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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this Department has been temporarily suspended.

A CERTAIN CLASS OF SEMIGROUPS

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Recently I had occasion to consider a certain cancellation semigroup S in which the equation $ab=xba$ is solvable for x when a and b are arbitrarily given in S. It is clear that every group is such a semigroup; so also is every subsemigroup of a group which contains the commutator subgroup of the given group. On the other hand it will be pointed out here that such semigroups although more general than groups have many properties of groups. This will be done in connection with the proof of our main theorem below and the remarks following it.

THEOREM. *If S is a cancellation semigroup in which the equation $ab=xba$ is solvable for x when a and b are arbitrarily given in S then S contains a subgroup G_* and S is contained in a super group G^* such that G^*/G_* is Abelian.*

Proof. We show first that S has a unique identity e. By assumption there is an e so that $aa=eaa$; it follows that $a=ea$, $ea=e^2a$ and $e=e^2$. Then for any b in S, $be=be^2$ and $b=be$; similarly $b=eb$. Hence e is an identity for S. If f were also an identity then $f=fe=e$ and hence e is unique.

Now if a and b are arbitrary in S then by assumption $ab=xba$. We denote this x by $[a, b]$ and call it the *commutator* of a with b. This commutator is unique; for $xba=yba$ implies $x=y$.

$[a, b]b$ will be called the *conjugate of b by a* and will be denoted by b^a.

Then N will be called a *normal subsemigroup* of S if N is a subsemigroup and if, for every n in N, n^s is also in N for every s in S.

If N is a normal subsemigroup of S then there is a homomorphism σ of S under which the set of elements mapped onto σe is precisely N. That is, there is a mapping σ so that if b is in S then σb is defined to be bN, the set of elements bn where n ranges over N; thus $\sigma b=bN$ and $\sigma(ab)=(\sigma a)(\sigma b)$. As usual bN will be called a coset of S with respect to N, and our assertion is that the set of cosets constitutes a semigroup denoted by S/N which is a homomorphic image of S. For the proof of the above statements we refer the reader to [2], pp. 33–34, the proof for groups being applicable.

We next show that the subsemigroup S' generated by all the commutators of S is a group. We have $ab=[a, b]ba$ and $ba=[b, a]ab$ whence $ab=[a, b][b, a]ab$ and $e=[a, b][b, a]$. Hence every commutator y has an inverse y^{-1}. Suppose $mn\cdots p$, and $rs\cdots t$ are elements of S' where m, n, p, r, s, and t are commutators; then the equation $mn\cdots p=x(rs\cdots t)$ has for solution $mn\cdots pt^{-1}\cdots s^{-1}r^{-1}$ which is in S', and hence S' is a group. S' will be the

group G_* mentioned in the theorem.

We will refer the reader to a criterion of Doss (*cf.* [1]), essentially a formulation of a result of Ore [3], for the proof that S can be imbedded in a group. The criterion of Doss requires that certain elements be regular on the left (a is regular on the left if for every b there are r and s so that $ra=sb$). Now in S we have that $ab=xba$ and hence every element is regular on the left; it follows that S can be imbedded in a group K. We will let G^* be the subgroup of K generated by the elements of S, and observe that every element of S has an inverse in G^*.

Now let $[a, b]$ be in G_* and let c be arbitrary in S. Then $[a, b]^c = caba^{-1}b^{-1}c^{-1} = caba^{-1}c^{-1}b^{-1}bcb^{-1}c^{-1} = [ca, b][b, c]$ is in G_* and hence G_* is transformed into itself by the elements of S. Then since the normalizer of S is a group (*cf.* [2] p. 26) it is clear that G_* is normal in G^*, the group generated by the elements of S. But now modulo G_* the generators of G^* commute with one another and hence G^*/G_* is Abelian as the theorem asserts. This completes the proof.

It is clear how one would define *solvable*, *nilpotent*, *simple*, *normal series* and so on for the above class of semigroups. In general one should have no difficulty in carrying over those theorems of group theory in which the ideas of conjugacy and normal subgroups play the major role. Also one notes that both simple and finite semigroups are groups and hence the distinction between composition series by groups and composition series by semigroups appears in the infinite Abelian factors.

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[1] Raouf Doss, Sur l'immersion d'un semi-groupe dans un groupe, Bulletin des Sciences Mathematiques, 2nd series, vol. 72, 1948, pp. 139–150.

[2] Hans Zassenhaus, Theory of groups, translated from the German, Chelsea, New York, 1949.

[3] Oystein Ore, Linear equations in non-commutative fields, Ann. Math., vol. 32, 1931, pp. 463–477.

THE MATRIC EQUATION $AX=B$ IN A FINITE FIELD

J. H. Hodges, Duke University

Let $q=p^n$ where p is a prime and let $GF(q)$ denote the Galois field of order q. Capital letters $A, B, \cdots$ will denote matrices with elements in $GF(q)$. $A(s, m)$ will denote a matrix of s rows and m columns and $A(s, m; \rho)$ a matrix of the same dimensions having rank ρ. $I(s, m; \rho)$ will denote the matrix of s rows and m columns having the identity of order ρ in its upper left-hand corner and zeros elsewhere. I_ρ will denote the identity of order ρ.

We seek the number $N(A, B)$ of solutions $X(m,t)$ of the equation

$$AX = B, \tag{1}$$

where $A=A(s, m; \rho)$ and $B=B(s, t; r)$ and $r \leqq \rho$. By [1, p. 281, Corollary 1] there exist non-singular P, Q, R, T such that $PAQ=I(s, m; \rho)$ and $RBT=$

$I(s,t;r)$. If we put $C=PR^{-1}=(\gamma_{ij})$, then (1) will reduce to

$$I(s, m; \rho)X = CI(s, t; r). \tag{2}$$

We will discuss the case when $\rho \leqq t$. By slight variations in the procedure, the same results are obtained when $\rho > t$. We write (2) as

$$\begin{bmatrix} I_\rho & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_{11} & X_{21} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} I(\rho, \rho; r) & 0 \\ 0 & 0 \end{bmatrix},$$

where $X_{11}=X_{11}(\rho,\ \rho)$, $X_{12}=X_{12}(\rho,\ t-\rho)$, $X_{21}=X_{21}(m-\rho,\ \rho)$, $X_{22}=X_{22}(m-\rho,\ m-\rho)$ and $C_{11}=C_{11}(\rho,\ \rho)$, $C_{12}=C_{12}(\rho,\ s-\rho)$, $C_{21}=C_{21}(s-\rho,\ \rho)$, $C_{22}=C_{22}(s-\rho,\ s-\rho)$. This implies that

$$\dot{X}_{11} = C_{11}\mathrm{I}(\rho, \rho; r); \qquad X_{12} = 0; \qquad X_{21}, X_{22} \quad \text{are arbitrary} \tag{3}$$

and also a necessary condition on C, namely $C_{21}I(\rho, \rho; r)=0$. Writing this condition as

$$[C^1_{21}\ \ C^2_{21}]\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = 0, \quad \text{where } C^1_{21} = C^1_{21}(s-\rho, r), \quad C^2_{21} = C^2_{21}(s-\rho, \rho-r),$$

we obtain the necessary condition $C^1_{21}=0$. Thus if $\gamma_{ij}=0$ for $\rho<i\leqq s$, $1\leqq j\leqq r$, the number of solutions of (2) is the number of choices of X_{21} and X_{22}. Thus we have proved

THEOREM 1. *If the equation* (1) *has any solutions, their number is*

$$N(A, B) = q^{t(m-\rho)}. \tag{4}$$

The condition for the existence of solutions is given by

THEOREM 2. *A necessary and sufficient condition that the equation* (1) *have solutions* $X(m, t)$ *is that there exist non-singular* P, Q, R, T *such that* $PAQ=I(s, m; \rho)$ *and* $RBT=I(s,t;r)$ *and if* $C=PR^{-1}=(\gamma_{ij})$, *then* $\gamma_{ij}=0$ *for* $\rho<i\leqq s$ *and* $1\leqq j\leqq r$.

The same method may be used to determine the number $N(A, B, C)$ of solutions $U=U(s, m)$, $V=V(t, n)$ of the matric equation

$$UA + BV = C, \tag{5}$$

where $A=A(m, n; r_1)$, $B=B(s, t; r_2)$ and $C=C(s, n; \rho)$. We are able to prove

THEOREM 3. *If the equation* (5) *has any solutions, their number is*

$$N(A, B, C) = q^{s(m-r_1)+n(t-r_2)+r_1r_2}. \tag{6}$$

A necessary and sufficient condition, similar to that given in Theorem 2, may be obtained for the existence of solutions of (5).

Reference

1. Garrett Birkhoff and Saunders MacLane, *A Survey of Modern Algebra*, New York, 1941.

PARTICULAR INTEGRALS OF LINEAR DIFFERENTIAL EQUATIONS

I. A. BARNETT, University of Cincinnati

In a recent issue of this MONTHLY (March, 1955, p. 174), Fettis develops a method for obtaining a particular integral of the linear differential equation with constant coefficients

$$Y^{(n)} + a_0 Y^{(n-1)} + \cdots + a_{n-1} Y = f(x).$$

The method given by the author is known as that of the Cauchy function and is explained in Goursat-Hedrick-Dunkel, *Differential Equations*, Vol. 2, Part II, p. 108. It applies equally well to equations with variable coefficients.

It is interesting to note that the Cauchy function described by Fettis is closely related to the resolvent kernel of an integral equation of the Volterra type (Goursat, *Cours d'Analyse Mathématique*, 5th ed. Vol. 3, pp. 309–21).

In fact, let

$$D(Y) \equiv Y^{(n)} - \lambda[a_0(x)Y^{(n-1)} + a_1(x)Y^{(n-2)} + \cdots + a_{n-1}(x)Y] = 0$$

be a linear differential equation of the nth order containing the parameter λ and having variable coefficients $a_i(x)$. If the function $K(x, y)$ defined by

$$K(x, y) \equiv a_0(x) + a_1(x)(x - y) + a_2(x)\frac{(x - y)^2}{2!} + \cdots + \frac{a_{n-1}(x)(x - y)^{n-1}}{(n - 1)!}$$

is the kernel of the integral equation

$$\phi(x) - \lambda \int_0^x K(x, s)\phi(s)ds = g(x),$$

and $\Gamma(x, y; \lambda)$ is the resolvent kernel of K, then

$$\Gamma(x, y; \lambda) = K(x, y) + \lambda \int_y^x \Gamma(x, s; \lambda)K(s, y)ds.$$

Let $G(x, y; \lambda)$ be the Cauchy function of $D(y) = 0$. Then the relation referred to above is

$$\Gamma(x, y; \lambda) = \frac{1}{\lambda}\frac{\partial^n G(x, y; \lambda)}{\partial x^n}.$$

If the coefficients a_i are constant, the preceding functions have the respective forms $K(x-y)$, $\Gamma(x-y; \lambda)$ and $G(x-y; \lambda)$. Thus a particular integral is $\int_0^x G(x-s)f(s)ds$ as given by Fettis. For the special equation treated, namely

$$Y^{iv} - 2Y'' + Y = f(x); \qquad (a = 1),$$

the kernel is $2(x-y) - \frac{1}{6}(x-y)$, and $\Gamma(x, y; 1) = \frac{1}{2}(x-y)\mathrm{ch}(x-y) + \frac{3}{2}\mathrm{sh}(x-y)$. The Cauchy function is

$$G(x-y) = (x-y)\operatorname{ch}(x-y) - \operatorname{sh}(x-y).$$

It may be verified that $\Gamma(x, y; 1) = G^{(iv)}(x, y; 1)$.

Finally, similar remarks may be made for Cauchy functions of simultaneous differential equations for which we would need to consider systems of Volterra equations.

UNIMODULAR COMPLEMENTS

IRVING REINER, Institute for Advanced Study and University of Illinois

A vector with rational integral components is called *primitive* if the greatest common divisor of its components is unity. A well-known theorem* (valid when the components lie in any principal ideal ring) states that any primitive vector can be completed to a unimodular matrix, that is, given any primitive row vector $\mathfrak{a}$, there exists a matrix of determinant ± 1 whose first row is $\mathfrak{a}$.

Due to the many applications of this theorem, it may be of interest to point out that the result holds in situations more general than that of a principal ideal ring, although the standard proof* depends strongly on the hypothesis that all ideals are principal. In 1911, Steinitz† developed the theory of elementary divisors for matrices over the ring R of algebraic integers in an algebraic number field. Even though R is not in general a principal ideal ring, his results imply that if $a_1, \cdots, a_n \in R$ generate the unit ideal in R, then there exists a matrix with first row $(a_1 \cdots a_n)$ whose determinant is a unit in R. The purpose of this note is to give a simple proof of this theorem, valid for any ring R in which classical ideal theory holds (see van der Waerden, *Modern Algebra* II, p. 83. New York, 1950, Ungar), and in particular valid for the case considered by Steinitz.

The theorem clearly holds for $n=1$. When $n=2$, we note that the ideal (a_1, a_2) generated by a_1 and a_2 in R consists of all sums $x_1a_1+x_2a_2$, $x_1 \in R$, $x_2 \in R$. Hence if $(a_1, a_2) = (1)$, there exist $b_1, b_2 \in R$ for which

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 1,$$

so the result holds for $n=2$.

Now let $n>2$, and assume the result established for a primitive k-component row vector, where $k<n$. Observe that if U is unimodular, then $\mathfrak{a} = (a_1 \cdots a_n)$ is completable if and only if $\mathfrak{a}U$ is completable; for if $\mathfrak{a}$ is the first row of the unimodular matrix T, then $\mathfrak{a}U$ is the first row of the unimodular matrix TU. Since U has an inverse, the argument also goes in the other direction.

Next we show there exists $b \in R$ for which $(a_1, \cdots, a_{n-2}, a_{n-1}-ba_n) = (1)$. For let $(a_1, \cdots, a_{n-2}) = \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_r^{\alpha_r}$ be the factorization of $(a_1, \cdots, a_{n-2})$

* MacDuffee, Theory of Matrices, Th. 21.1, p. 31. Berlin 1933, Springer.

† Math. Ann., Vol. 71, 1911, pp. 328–354.

into powers of distinct prime ideals. We need only choose b so that

$$a_{n-1} \not\equiv ba_n \pmod{\mathfrak{p}_i} \qquad (i = 1, \cdots, r).$$

For any $\mathfrak{p}_i$ dividing a_n, this holds for all b, since such a $\mathfrak{p}_i$ does not divide a_{n-1}. For the $\mathfrak{p}_i$ not dividing a_n, the congruence $a_n x \equiv a_{n-1} \pmod{\mathfrak{p}_i}$ has a unique solution x_i. For these $\mathfrak{p}_i$, we impose the condition

$$b \equiv x_i + 1 \pmod{\mathfrak{p}_i}.$$

These congruences can be solved for b by the Chinese Remainder Theorem, which is certainly valid under the hypotheses on R (see van der Waerden, *op. cit.*, II, p. 41).

Now we have

$$(a_1 \cdots a_{n-1} \quad a_n) \begin{pmatrix} I^{(n-2)} & & 0 \\ & 1 & 0 \\ 0 & -b & 1 \end{pmatrix} = (a_1 \cdots a_{n-2} \quad a_{n-1} - ba_n \quad a_n),$$

with the above matrix unimodular. Furthermore, by the induction hypothesis, $(a_1 \cdots a_{n-2}\ a_{n-1} - ba_n)$ is completable to a unimodular matrix V. Hence $(a_1 \cdots a_{n-2}\ a_{n-1} - ba_n\ a_n)$ is completable to

$$\begin{pmatrix} & & a_n \\ & V & 0 \\ & & \vdots \\ 0 \cdots & 0 & 1 \end{pmatrix}.$$

This completes the proof.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE IRRATIONALITY OF $\sqrt{2}$

ROBERT GAUNTT, Purdue University, and GUSTAVE RABSON, Antioch College

(The following proof was invented by Robert James Gauntt, in 1952, while he was a freshman at Purdue. I was unable to induce him to write up his proof. G. R.)

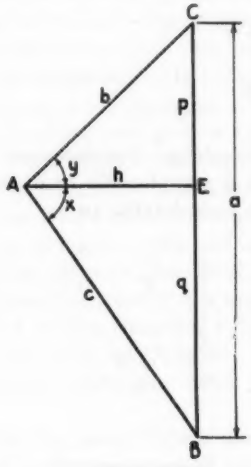
$a^2 = 2b^2$ cannot have a non-zero solution in integers because the last non-zero digit of a square, written in the base three, must be 1, whereas the last non-zero digit of twice a square is 2.

ADDITION THEOREMS FOR THE SINE AND COSINE

CARL COHEN, Cambridge Junior College

If the angles x and y are both positive and acute, the following construction leads to simple proofs of the addition theorems for the sine and cosine. We assume a knowledge of the formula for the area of a triangle in terms of two sides and the included angle, as well as the law of the cosine.

We draw angles x, y, and $x+y$ with common vertex A. Since x and y are positive acute angles, a line perpendicular to the common edge AE of x and y at E will intersect the other edges of x and y as indicated in the figure, at points which we call B and C respectively.



In this figure, the area S of ABC is:

(1) $$S_{ABC} = \tfrac{1}{2}bc \sin (x + y).$$

On the other hand, by construction,

(2) $$S_{ABC} = S_{ACE} + S_{ABE} = \tfrac{1}{2}bh \sin y + \tfrac{1}{2}ch \sin x.$$

Equating (1) and (2), we obtain

$$bc \sin (x + y) = bh \sin y + ch \sin x,$$

and, after dividing by bc,

$$\sin (x + y) = \frac{h}{c} \sin y + \frac{h}{b} \sin x = \sin y \cos x + \sin x \cos y.$$

Applying the law of the cosine, we write

$$a^2 = b^2 + c^2 - 2bc \cos (x + y). \tag{3}$$

Again, by construction, we have $a = p + q$, therefore

$$a^2 = p^2 + 2pq + q^2. \tag{4}$$

After applying the theorem of Pythagoras to both right triangles, we change (4) into

$$a^2 = b^2 + 2pq - 2h^2 + c^2. \tag{5}$$

Now, we equate (3) and (5) and obtain:

$$b^2 + 2pq - 2h^2 + c^2 = b^2 + c^2 - 2bc \cos (x + y).$$

We cancel b^2 and c^2 and divide both sides by $2bc$. The result is:

$$\cos (x + y) = \frac{h}{b} \cdot \frac{h}{c} - \frac{p}{b} \cdot \frac{q}{c},$$

which leads immediately to the wanted formula.

AN AVERAGING METHOD OF EXTRACTING ROOTS

J. P. BALLANTINE, University of Washington

1. Statement of the averaging method. To find the n-th root of any positive number, W, write

$$W = x^n = x_1^{n-1} x_2, \tag{1}$$

where x is the value sought, x_1 is an approximation to x, and x_2 is found by dividing W by x_1^{n-1}.

The next approximation, $\bar{x}$, is found by taking the weighted average of x_1 and x_2. The usual practice is to give x_1 and x_2 weights $n-1$ and 1, the same as their exponents in equation (1). By making the weights depend on the values of x_1 and x_2, better results may be had. It is the purpose of this paper to work out such weights, when the dependence is linear.

2. Derivation of the weights. Let

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}, \text{ where} \tag{2}$$

$$w_1 = w_{11} x_1 + w_{12} x_2, \text{ and} \tag{3}$$

$$w_2 = w_{21} x_1 + w_{22} x_2.$$

The constants w_{11}, w_{12}, w_{21}, and w_{22} will now be determined in such a way as to make $\bar{x}$ as nearly equal to x as possible.

Let

$$x_1 = x(1 + a).$$

Then

$$x_2 = x(1+a)^{1-n}$$
$$= x\left(1 + (1-n)a + \frac{-n(1-n)}{2}a^2 + \frac{n(1-n^2)}{6}a^3 \cdots\right).$$

We are, of course, supposing that a is small, so that the binomial expansion converges. Similar series are easily written for x_1^2, x_1x_2, and x_2^2.

In equation (2), the letters w_1 and w_2 are replaced by their values in equation (3), and now the letters x_1 and x_2 are replaced by their values in terms of x, a, and n. The result will be a fraction, with both numerator and denominator expressed in powers of a, namely:

$$\bar{x} = \frac{x^2(U_0 + U_1a + U_2a^2 + \cdots)}{x(V_0 + V_1a + V_2a^2 + \cdots)}, \tag{4}$$

where

$$U_0 = V_0 = w_{11} + w_{12} + w_{21} + w_{22}.$$
$$U_1 = 2w_{11} + (2-n)w_{12} + (2-n)w_{21} + (2-2n)w_{22}.$$
$$V_1 = w_{11} + w_{21} + (1-n)w_{12} + (1-n)w_{22}.$$
$$U_2 = w_{11} + \tfrac{1}{2}(n-1)(n-2)(w_{12} + w_{21}) + (n-1)(2n-1)w_{22}.$$
$$V_2 = \tfrac{1}{2}n(n-1)(w_{12} + w_{22}).$$
$$U_3 = \tfrac{1}{6}(n)(n-1)(2-n)(w_{12} + w_{21}) + \tfrac{2}{3}n(2n-1)(1-n)w_{22}.$$
$$V_3 = \tfrac{1}{6}n(1-n)(1+n)(w_{12} + w_{22}).$$

In order for x and $\bar{x}$ to be nearly equal, the parentheses in equation (4) must be nearly equal. Fortunately $U_0 = V_0$, without restriction on the weights. Though we have 4 weights, only the ratios between them count, so we may expect to impose 3 conditions, namely $U_1 = V_1$, $U_2 = V_2$, and $U_3 = V_3$. These are three homogeneous linear equations among the four weights, w_{ij}. Their solutions are:

$$w_{11} = (2n-1)(n-1), \qquad w_{12} = (n-1)(4n+1),$$
$$w_{21} = 5n - 1, \qquad w_{22} = n + 1. \tag{5}$$

3. Square roots. When $n = 2$, a factor 3 drops out, and

$$w_{11} = 1, \quad w_{12} = 3, \quad w_{21} = 3, \quad \text{and} \quad w_{22} = 1. \tag{6}$$

For example, to find $\sqrt{30}$, take $x_1 = 5$ and $x_2 = 6$. Then 5 has a weight of $5 + 3 \cdot 6 = 23$ and 6 has a weight of $6 + 3 \cdot 5 = 21$. The weighted average is

$$\frac{23 \cdot 5 + 21 \cdot 6}{23 + 21} = \frac{241}{44} = 5.4772727 \cdots.$$

Actually,

$$\sqrt{30} = 5.4772255\cdots.$$

The usual $\frac{1}{2}(5+6)=5.5$ gives a much poorer approximation.

For the special case when $n=2$, formula (2) can be put into a more convenient form for computation, by substituting w_1 and w_2 from equation (3) into formula (2).

$$\bar{x} = \frac{w_{11}x_1^2 + (w_{12}+w_{21})x_1x_2 + w_{22}w_2^2}{(w_{11}+w_{21})x_1 + (w_{12}+w_{22})x_2},$$

$$\bar{x} = \frac{4x^2 + (x_1+x_2)^2}{4(x_1+x_2)}. \tag{7}$$

For a direct derivation of formula (7) with remainder, write:

$$x - \tfrac{1}{2}(x_1 - x_2) = e$$

$$2x - (x_1 + x_2) = 2e$$

$$4x^2 - 4x(x_1+x_2) + (x_1+x_2)^2 = 4e$$

$$x = \frac{4x^2 + (x_1+x_2)^2}{4(x_1+x_2)} - \frac{e^2}{x_1+x_2}. \tag{8}$$

For example, suppose the approximation 1.4 to $\sqrt{2}$ is known. Then $x_1=1.4$, $x_2=1.428571428\cdots$ by division, and $x^2=2$.

$$\bar{x} \approx \frac{4(2)+(2.828571428)^2}{4(1.4+1.428571428)} = \frac{16.0008163265}{11.3142857143} = 1.414213564.$$

To approximate e of formula (8), write

$$e \approx \bar{x} - \tfrac{1}{2}(x_1+x_2) = 1.4142135 - 1.4142857 \approx -0.000072$$

$$\frac{e^2}{x_1+x_2} \approx \frac{0.0000000052}{2.8} = 0.000000002.$$

So that

$$x \approx \bar{x} - 0.000000002 = 1.414213562.$$

4. Cube roots. When $n=3$, a factor 2 drops out, and

$$w_{11} = 5,\ w_{12} = 13,\ w_{21} = 7, \quad \text{and} \quad w_{22} = 2. \tag{9}$$

For example, to find $\sqrt[3]{100}$, write $5^2\cdot 4=100$. Then 5 has a weight $5\cdot 5+13\cdot 4=77$, and 4 has a weight of $7\cdot 5+2\cdot 4=43$. The weighted average is

$$\bar{x} = \frac{77\cdot 5 + 43\cdot 3}{77+43} = \frac{557}{120} = 4.641666\cdots.$$

This compares with the correct answer, $\sqrt[3]{100}=4.6415888\cdots$, and the usual average of 5 and 4 with weights 2 and 1 of $4.666\cdots$.

Referee's Note. The method given is a fourth-order iterative process for the determination of an nth root. It happens that the iteration formula is identifiable with the third of a sequence of such formulas obtainable as successive convergents of the formal continued fraction

$$\bar{x} = x_1 - \cfrac{\omega}{1 - \cfrac{(n-1)\omega/(2x_1)}{1 - \cfrac{(n+1)\omega/(6x_1)}{1 - \cdot}}}$$

where

$$\omega = \frac{1}{n}\left(x_1 - \frac{W}{x_1^{n-1}}\right) \equiv \frac{1}{n}(x_1 - x_2),$$

with the author's notation, the two preceding formulas consequently being of the forms

$$\bar{x}^{(1)} = x_1 - \omega = \frac{1}{n}[(n-1)x_1 + x_2]$$

and

$$\bar{x}^{(2)} = x_1 - \cfrac{\omega}{1 - \cfrac{n-1}{2}\cfrac{\omega}{x_1}} = \frac{(n-1)x_1 + (n+1)x_2}{(n+1)x_1 + (n-1)x_2}x_1,$$

of second and third order, respectively.

The definition of $\bar{x}^{(1)}$ corresponds to the Newton-Raphson iteration, as applied to $f(x)=x^n-W$, while the approximation $\bar{x}^{(2)}$ corresponds to the so-called Halley iteration, as applied to the same function. For the generation of the sequence, see Frame (this MONTHLY, vol. 60, 1953, pp. 293–305).

In the special case $n=2$, the iteration (7) is equivalent to two successive applications of the Newton-Raphson iteration $\bar{x}^{(1)}$.

Whereas, for the purpose of minimizing the total labor, the usual tendency is to use a second-order process twice rather than one of the fourth order processes once, the author's calculational arrangement is a simple one and it may find some favor in practice.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1211. *Proposed by Alan Wayne, Cooper Union School of Engineering*

Prove the following theorem, discovered by Cipriano Ferraris of the *New York Riddlers Club:* "For any positive integer k, the total number of digits in the sequence $1, 2, \cdots, 10^k$ equals the number of zero digits in the sequence $1, 2, \cdots, 10^{k+1}$."

E 1212. *Proposed by H. A. Osborn, University of California, Berkeley*

Show that $t>0$ implies $(2+\cos t)t>3\sin t$.

E 1213. *Proposed by F. D. Parker, Clarkson College of Technology*

Find the Wronskian, $W(n, k)$, of the set of functions $x^n, x^n \ln x, x^n \ln^2 x, \cdots, x^n \ln^k x$.

E 1214. *Proposed by Paul Payette, École Polytechnique de Montreal*

Find the envelope of the family of ellipses of constant major axis having one focus at a given point and the other focus on a given straight line.

E 1215. *Proposed by J. P. Ballantine, University of Washington*

State a necessary and sufficient condition for an ordered set of n line segments to be the consecutive sides of an n-gon possessing an inscribed circle.

SOLUTIONS

E 1171 [1956, 121]. *Correction.* The *Editorial Note* should be corrected to read " . . . , then $(n^a-1, n^b-1)=n-1$ if and only if $(a, b)=1$."

Divisors of a String of Ones

E 1181 [1955, 581]. *Proposed by F. L. Wolf, Carleton College*

Show that if an integer has all its prime factors greater than 5 then it divides some integer whose decimal representation is a string of 1's.

Solution by W. E. Briggs, University of Colorado. Since $(n, 10)=1$, n divides $10^{\phi(n)}-1=9(111\cdots 1)$. Since $(n, 3)=1$, n divides the factor consisting only of ones.

Also solved by J. A. Bond, Jr., D. H. Browne, R. L. Causey, G. B. Charlesworth, Fred Cherry, P. A. Clement, A. E. Danese, M. Delcourte, Calvin Foreman, Michael Goldberg, A. J. Goldman, D. S. Greenstein, C. A. Grimm, Nathaniel Grossman, Cornelius Groenewoud, Virginia Hanly, Juris Hartmanis, B. A. Hausmann, M. J. Hellman, Vern Hoggatt, C. W. Langley, D. C. B. Marsh, J. D. Miller, Leo Moser, T. F. Mulcrone, J. B. Muskat, C. S. Ogilvy, Paul Payette, Walter Penney, L. L. Pennisi and N. C. Scholomiti (jointly), Azriel Rosenfeld, David Rothman, C. M. Sandwick, Sr., J. P. Scholz, R. E. Shafer, G. J. Simmons, E. P. Starke, R. P. Tapscott, C. W. Topp, Chih-yi Wang, R. J. Wisner, F. S. Zusman, and the proposer. Late solutions by J. G. Clunie, F. I. John, M. S. Klamkin, and C. F. Pinzka.

An Identity

E 1182 [1955, 581]. *Proposed by I. A. Barnett, University of Cincinnati*

If $d_k = u^k - x^k - y^k - z^k$, for $k = 1, 2, 3, \cdots$, establish the identity

$$d_{n+2} - (x + y + z)d_{n+1} + (xy + yz + zx)d_n - (xyz)d_{n-1} = u^{n-1}(u - x)(u - y)(u - z).$$

Solution by J. A. Tierney, U. S. Naval Academy. Denote the left member of the proposed identity by $f(u)$. A simple calculation shows that $f(x) = f(y) = f(z) = 0$. Hence $f(u)$ has $u-x$, $u-y$, $u-z$ as factors. In fact, $f(u) = u^{n-1}(u-x)(u-y)\cdot(u-z)$, since the four coefficients of like powers of u are equal by inspection.

Also solved by A. N. Aheart, Norman Anning, G. B. Charlesworth, A. E. Danese, J. E. Darraugh, M. Delcourte, H. M. Feldman, Edward Fleisher, Calvin Foreman, A. J. Goldman, D. S. Greenstein, C. A. Grimm, Nathaniel Grossman, M. J. Hellman, Vern Hoggatt, A. R. Hyde, P. W. M. John, D. R. Lewis, D. C. B. Marsh, W. R. McEwen, J. B. Muskat, Herbert Nadler, A. R. Nolstad, Walter Penney, B. E. Rhoades, James Robertson and Dale Woods (jointly), D. A. Robinson, Azriel Rosenfeld, David Rothman, R. E. Shafer, O. E.Stanaitis, E. P. Starke, Chih-yi Wang, Charlotte Yesselman, David Zeitlin, and the proposer. Late solutions by J. E. D'Atri, Hüseyin Demir, F. I. John, M. S. Klamkin, C. F. Pinzka, K. Subba Rao, and D. C. Russell.

Editorial Note. Denote the left member of

$$r^{n+2} - (x + y + z)r^{n+1} + (xy + yz + zx)r^n - xyzr^{n-1} \equiv r^{n-1}(r - x)(r - y(r - z)$$

by $g(r)$. The left member of the proposed identity is then given by $g(u) - g(x) - g(y) - g(z)$, or, since $g(x) = g(y) = g(z) = 0$, by $g(u)$, which establishes the proposed identity.

A Simple Series Summation

E 1183 [1955, 581]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Prove that $\sum_{n=1}^{\infty} (n-1)/n! = 1$.

I. *Solution by H. M. Gehman, University of Buffalo.* Let $\sum$ denote summation from $n=1$ to $n=k$. Then $S=\sum(n-1)/n!=\sum 1/(n-1)!-\sum 1/n!=1-(1/k!)$. As k becomes infinite, S approaches 1.

This problem occurs in the *Mathematics Student Journal* for October, 1955, p. 4, problem 46. It is attributed to the eighth annual Stanford University Competitive Examination, April 11, 1943.

II. *Solution by Edward Fleisher, Brooklyn, N. Y.* Since the series converges absolutely, we have

$$\sum_{n=1}^{\infty}(n-1)/n! = \sum_{n=1}^{\infty} 1/(n-1)! - \sum_{n=1}^{\infty} 1/n! = e-(e-1) = 1.$$

Also solved by W. A. Al-Salam, Ferrel Atkins, G. E. Bardwell, P. M. Berry, Nardler Berseck, W. E. Briggs, J. L. Brown, Jr., R. L. Causey, G. B.Charlesworth, George Cherlin, P. L. Chessin, A. E. Danese, J. E. Darraugh, M. Delcourte, Thomas Erber, H. M. Feldman, Allen Fenstermacher, Walter Fleming and David Lewis (jointly), W. L. Fields, Calvin Foreman, David Freedman, Joyce Friedman, B. A. Fusaro, A. M. Glicksman, Michael Goldberg, A. J. Goldman, D. S. Greenstein, C. A. Grimm, Cornelius Groenewoud, Nathaniel Grossman, Peter Hagis, Jr., W. G. Hanks, Frank Harary, Juris Hartmanis, M. J. Hellman, Vern Hoggatt, F. A. Homann, Raymond Huck, A. R. Hyde, L. E. Isenecker, P. W. M. John, J. D. E. Konhauser, M. A. Kirchberg, P. G. Kirmser, W. D. Lambert, C. W. Langley, Kenneth Loewen, P. K. Maloof, D. C. B. Marsh, L. C. Marshall, W. R. McEwen, G. M. Merriman, Erich Michalup, J. D. Miller, Leo Moser, T. F. Mulcrone, J. B. Muskat, A. R. Nolstad, C. S. Ogilvy, Hyman Orlin, F. D. Parker, M. J. Pascual, Paul Payette, W. O. Pennell, Walter Penney, L. L. Pennisi and N. C. Scholomiti (jointly), W. J. Pervin, J. P. Phillips, R. B. Plymale, B. E. Rhoades, L. A. Ringenberg, James Robertson and Dale Woods (jointly), D. A. Robinson, Azriel Rosenfeld, J. W. Ross, David Rothman, C. M. Sandwick, Sr., Nina Schub, R. E. Shafer, W. A. Small, D. W. Smith, O. E. Stanaitis, E. P. Starke, F. M. Stein, D. D. Strebe, R. P. Tapscott, C. W. Topp, Chih-yi Wang, Alan Wayne, R. R. Williams, Jr., R. L. Wilson, David Zeitlin, F. S. Zusman, and the proposer. Late solutions by Hüseyin Demir, R. F. Gabriel, A. G. Grace, Jr., F. I. John, M. S. Klamkin, D. J. Peterson, C. F. Pinzka, K. Subba Rao, D. C. Russell, and Bernard Smilowitz.

Evaluation of a Limit

E 1184 [1955, 581]. *Proposed by Viktors Linis, University of Ottawa*

Let $f(x)$ be differentiable at $x=a$ and let $f(a)\neq 0$. Evaluate

$$\lim_{n\to\infty}[f(a+1/n)/f(a)]^n.$$

Solution by R. F. Church, Nicolas Johnson, and F. J. Lorenzen, Prof. R. M. Conkling's Advanced Calculus Class, University of New Hampshire. The limit sought is the same as

$$\lim_{x\to 0} [f(a+x)/f(a)]^{1/x}.$$

For x small enough, $f(a+x)$ and $f(a)$ have the same sign. Then

$$\lim_{x\to 0} \log [f(a+x)/f(a)]^{1/x} = \lim_{x\to 0} [\log f(a+x) - \log f(a)]/x.$$

Since a differentiable function of a differentiable function is differentiable, this last limit is, by definition, the derivative of $\log f(x)$ at $x=a$, or $f'(a)/f(a)$. Hence

$$\lim_{x\to 0} [f(a+x)/f(a)]^{1/x} = \exp [f'(a)/f(a)].$$

Also solved by W. A. Al-Salam, Nardler Berseck, W. E. Briggs, G. B. Charlesworth, P. L. Chessin, A. E. Danese, M. Delcourte, Thomas Erber, H. M. Feldman, Walter Fleming, David Freedman, B. A. Fusaro, Michael Goldberg, A. J. Goldman, D. S. Greenstein, Nathaniel Grossman, Peter Hagis, Jr., Virginia Hanly, Juris Hartmanis, R. E. Heaton, M. J. Hellman, Vern Hoggatt, A. R. Hyde, L. E. Isenecker, P. G. Kirmser, D. C. B. Marsh, L. C. Marshall, J. D. Miller, Morris Morduchow, Leo Moser, A. R. Nolstad, C. S. Ogilvy, M. J. Pascual, Paul Payette, Walter Penney, H. P. Rawlings, L. A. Ringenberg, Azriel Rosenfeld, J. W. Ross, David Rothman, W. A. Small, O. E. Stanaitis, E. P. Starke, F. M. Stein, D. D. Strebe, R. P. Tapscott, Chih-yi Wang, J. V. Whittaker, David Zeitlin, and the proposer. Late solutions by Hüseyin Demir, A. G. Grace, Jr., F. I. John, M. S. Klamkin, and D. C. Russell.

The Proposer pointed out that this problem permits the construction of interesting examples, such as

$$\lim_{n\to\infty} \tan^n(\pi/4 + 1/n) = e^2$$

and

$$\lim_{n\to\infty} [\cos(a + 1/n)/\cos a]^n = e^{-\tan a}, \qquad \cos a \neq 0.$$

Infinite Monotone Subsequences

E 1185 [1955, 581]. *Proposed by Frederic Cunningham, Jr., University of New Hampshire.*

Prove that every infinite sequence of real numbers contains an infinite monotone subsequence.

I. *Solution by A. J. Goldman, Princeton University.* Let $\{x_n\}$ be an infinite sequence of real numbers. If, on the one hand, every infinite subsequence of $\{x_n\}$ has a (strictly) largest term, then let y_1 be the largest term of $\{x_n\}$, let y_2 be the largest term of the subsequence of $\{x_n\}$ consisting of all terms of $\{x_n\}$ *after* y_1, *etc.* Then $\{y_1, y_2, \cdots\}$ is a (strictly) monotone decreasing subsequence of $\{x_n\}$. If, on the other hand, some infinite subsequence $\{u_i\}$ of $\{x_n\}$ does *not* have a strictly largest term, then we set $z_1 = u_1$, choose z_2 to be a term *after* z_1 in $\{u_i\}$ which is no smaller than z_1, *etc.* Then $\{z_1, z_2, \cdots\}$ is an infinite monotone ascending subsequence of $\{x_n\}$.

The proof uses only order-theoretic properties of the real numbers, and is valid in any totally-ordered set.

II. *Solution by Azriel Rosenfeld, Columbia University.* If the sequence is unbounded, say to the right, the construction of a strictly ascending subsequence is immediate. Similarly, unboundedness to the left yields a strictly descending subsequence.

If the sequence is bounded, it has a cluster point (in the generalized sense in which the same point, counted infinitely often in the sequence, is a cluster point of the sequence too). Then an infinite number of terms of the sequence are contained in some (small, half-closed) interval either to the right or to the left of this point. The construction respectively of a nondecreasing or a nonincreasing subsequence is then clear.

III. *Solution by Leo Moser, University of Alberta.* This is an immediate consequence of the following theorem due to P. Erdös and G. Szekeres (*A combinatorial problem in geometry*, Composition Mathematica, vol. 2, 1935, pp. 463–470):

Theorem. From n points with monotonically increasing abscissae it is always possible to select at least $\sqrt{n}$ points with increasing abscissae and either monotonically increasing or decreasing ordinates. (If two ordinates are equal, the case may equally be regarded as increasing or decreasing.)

Also solved by A. E. Danese, D. S. Greenstein, Virginia Hanly, Juris Hartmanis, Vern Hoggatt, A. R. Hyde, L. E. Isenecker, C. W. Langley, Norman Levine, D. C. B. Marsh, Paul Payette, L. A. Ringenberg, J. W. Ross, David Rothman, J. P. Scholz, W. L. Shepherd, O. E. Stanaitis, E. P. Starke, D. D. Strebe, R. P. Tapscott, R. J. Wisner, and the proposer. Late solutions by Nathaniel Grossman, M. S. Klamkin, C. F. Pinzka, and D. C. Russell.

Editorial Note. This problem provides a direct method of proving that a Cauchy sequence has a limit. See, *e.g.*, Dick Wick Hall and Guilford L. Spencer II, *Elementary Topology* (John Wiley and Sons, 1955), pp. 30–32.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4683. *Proposed by Edgar Karst, Independence, Missouri*

Where fails the "interesting conjecture" (so called by D. H. Lehmer in a letter to the proposer, March 1952): If 2^p-1 is prime, then also $2^p-1+100$ is prime?

4684. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Prove that $a_n \to 0$ as $n \to \infty$, where

$$a_n = 1 - \frac{n-1}{1!} + \frac{(n-2)^2}{2!} - \frac{(n-3)^3}{3!} + \cdots + \frac{1}{(n-1)!}.$$

4685. *Proposed by L. J. Mordell, University of Toronto*

Prove that the n-dimensional polytope

$$|x_1| + |x_2| + \cdots + |x_n| + |x_1 + x_2 + \cdots + x_n| \leqq 2,$$

has $2^{n+1}-2$ faces of $n-1$ dimensions and contains an n-dimensional volume of $(2n)!/(n!)^3$.

4686. *Proposed by Leonard Carlitz, Duke University*

Show that

$$\lim_{n\to\infty} \left\{ \sum_{k=1}^{n} (2k-1) \tanh \frac{(2k-1)\pi}{2} - n^2 \right\} = -1/12, \tag{1}$$

$$\lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k \coth k\pi - \frac{n(n+1)}{2} \right\} = 1/12 - 1/4\pi. \tag{2}$$

4687. *Proposed by John Wermer, Brown University*

Let Γ be a simple closed curve in the complex plane containing the origin in its interior. Show that if Γ is not rectifiable, then we can approximate the constant 1 uniformly on Γ by functions

$$\sum_{n=-N}^{N} c_n z^n, \qquad c_0 = 0.$$

SOLUTIONS

An Indeterminate Form

4622 [1955, 45]. *Proposed by Harry Furstenberg, New York City*

Let r be any positive integer and let $i = r, r+1, \cdots, n$ and $j = 1, 2, \cdots, r$. Prove

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j} \log\left(1 - \frac{i}{n+j}\right) = -r.$$

I. *Solution by Vladeta Vučković, Mathematical Institute, Belgrade, Yugoslavia.*

Direct computation and application of Stirling's formula give

$$\frac{1}{n}\sum_{j=1}^{r} \log\left(1 - \frac{r}{n+j}\right)\left(1 - \frac{r+1}{n+j}\right)\cdots\left(1 - \frac{n}{n+j}\right)$$

$$= \sum_{j=1}^{r} \log \sqrt[n]{\frac{(n-r+j)!}{(j-1)!(n+j)^{n-r+i}}} \sim \sum_{j=1}^{r} \log \frac{1}{e} = -r.$$

II. *Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota.* If we write

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j} \log\left(1 - \frac{i}{n+j}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=r}^{n} \log\left(1 - \frac{i}{n+1}\right)$$

$$+ \lim_{n\to\infty} \frac{1}{n} \sum_{i=r}^{n} \log\left(1 - \frac{i}{n+2}\right) + \cdots + \lim_{n\to\infty} \frac{1}{n} \sum_{i=r}^{n} \log\left(1 - \frac{i}{n+r}\right),$$

and make use of the definition of the definite integral, it follows at once that each of the r terms on the right side is equal to the familiar integral

$$\int_0^1 \log(1-x)\,dx = -1.$$

This completes the proof.

Also solved by A. J. Goldman, M. S. Klamkin, Julius Lieblein, A. E. Livingston, L. A. Ringenberg, Chih-yi Wang, J. V. Whittaker, and the Proposer.

A Variant of the Probability Integral

4623 [1955, 126]. *Proposed by L. L. Pennisi, University of Illinois, Navy Pier, Chicago*

Prove that

$$\lim_{x\to\infty} \frac{\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{3\cdot 5\cdot\cdots\cdot(2n+1)}}{\sum_{n=0}^{\infty} \dfrac{x^{2n}}{2\cdot 4\cdot\cdots\cdot 2n}} = \sqrt{\frac{\pi}{2}}.$$

Solution by Emil Grosswald, University of Pennsylvania.

Let $\phi(x)=f(x)/g(x)$, with $f(x)=\sum x^{2n+1}/3\cdot 5\cdot\cdots\cdot(2n+1)$, and $g(x)=\sum x^{2n}/2\cdot 4\cdot\cdots\cdot 2n$. It is easily seen that $f(x)$ and $g(x)$ are entire functions, satisfying the differential equations and initial conditions

$$f'(x) - xf(x) - 1 = 0 \qquad f(0) = 0,$$

$$g'(x) - xg(x) = 0 \qquad g(0) = 1.$$

The solutions by familiar methods are

$$f(x) = e^{x^2/2}\int_0^x e^{-t^2/2}dt, \qquad g(x) = e^{x^2/2}.$$

Consequently $\phi(x)=\int_0^x e^{-t^2/2}dt$, the classical probability integral, and the desired limit is $\sqrt{\pi/2}$.

Also solved by W. J. Blundon, B. J. Boyer, G. U. Brauer and Chih-yi Wang, Robert Breusch, F. A. Butter, Jr., G. B. Findley, N. J. Fine, Harley Flanders, A. J. Goldman, S. W. Golomb, W. O. Gordon, Cornelius Groenewoud, Emil Grosswald, Peter Hagis, Jr., Peter Henrici, A. R. Hyde, M. S. Klamkin, J. A. Larrivee, David Lovenvirth, L. I. Lowell, S. Marein-Efron, D. C. B. Marsh, C. S. Ogilvy, F. R. Olson, W. J. Pervin, G. Pólya, E. D. Rainville, L. A. Ringenberg, P. G. Rooney, D. C. Russell, F. C. Smith, M. R. Spiegel, T. W. Summers R. P. Tapscott, Arnold Walfisz, the Proposer, and one whose name does not appear. Late solution by O. E. Stanaitis.

Editorial Note. Pólya notes that the essential part of the problem appeared on the Putnam Prize examination for 1950. See [1950, 469].

A Beta-Function

4625 [1955, 127]. *Proposed by K.-F. Moppert, Basel, Switzerland*

For $a>0$, $b>-1$, prove

$$\frac{1}{a} - \binom{b}{1}\frac{1}{1+a} + \binom{b}{2}\frac{1}{2+a} - + \cdots = \frac{\Gamma(a)\Gamma(1+b)}{\Gamma(1+a+b)}.$$

I. *Solution by M. R. Spiegel, Rensselaer Polytechnic Institute.* We have

$$\int_0^x u^{a-1}(1-u)^b du = \int_0^x u^{a-1}\left[1 - \binom{b}{1}u + \binom{b}{2}u^2 - \cdots\right]du$$

$$= \frac{x^a}{a} - \binom{b}{1}\frac{x^{a+1}}{1+a} + \binom{b}{2}\frac{x^{a+2}}{2+a} - \cdots$$

if $0<x<1$ and $b>-1$. Letting $x\to 1$ and employing Abel's theorem it follows that

$$\int_0^1 u^{a-1}(1-u)^b du = \frac{\Gamma(a)\Gamma(1+b)}{\Gamma(1+a+b)} = \frac{1}{a} - \binom{b}{1}\frac{1}{1+a} + \binom{b}{2}\frac{1}{2+a} - \cdots,$$

which proves the result. If b is a positive integer or zero the series terminates and Abel's theorem is unnecessary. The uniform convergence of $(1-x)^b$ in the interval (0, 1) justifies the interchange of summation and integration.

II. *Solution by E. D. Rainville, University of Michigan.* The restrictions a not a nonpositive integer and $\mathrm{Re}(b)>-1$ are sufficient. Let ψ denote the series to be summed:

$$\psi = \sum_{n=0}^{\infty} \frac{(-1)^n\binom{b}{n}}{a+n} = \frac{1}{a}\sum_{n=0}^{\infty}\frac{(-b)_n(a)_n}{(a+1)_n n!} = \frac{1}{a}\,{}_2F_1(-b, a; a+1; 1).$$

If $\mathrm{Re}(b)>-1$, then

$$\psi = \frac{1}{a}\,\frac{\Gamma(a+1)\Gamma(a+1+b-a)}{\Gamma(a+1-a)\Gamma(a+1+b)} = \frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b+1)}.$$

Also solved by P. M. Batchelder, B. J. Boyer, Leonard Carlitz, A. E. Danese, G. B. Findley, C. L. Gape, A. J. Goldman, V. C. Harris, Peter Henrici, A. R. Hyde, M. S. Klamkin, S. Marein-Efron, F. H. Northover, C. D. Olds, P. G. Rooney, D. C. Russell, N. Shklov, F. C. Smith, T. W. Summers, and Chih-yi Wang.

Editorial Note. The problem may be found in several places, *e.g.*, Gibson, *Advanced Calculus*, p. 498, no. 23; Bromwich, *Infinite Series*, p. 194, ex. 31.

Point Set Dense in the Plane

4626 [1955, 127]. *Proposed by W. A. Michael and D. A. Page, University of Illinois*

Let A be a subset of the plane. Join each pair of distinct points of A by a line. Let A_1 consist of the points of A together with the points of intersection of distinct lines. Starting with A_1 the same construction yields a set A_2. In this way an increasing sequence of sets $A_1, A_2, \cdots$, is obtained. Let $\Omega=\cup A_n$. If A

is denumerable then so is Ω, and hence Ω is not the entire plane; however, under what conditions will Ω be dense in the plane?

Solution by the Proposers. Ω will be dense in the plane if and only if there are four points in A such that

(i) no three are collinear, and

(ii) the four points are not the vertices of a parallelogram.

For suppose A contains four such points p_1, p_2, p_3, p_4. Exactly two possibilities arise: *Case* 1. One of the points, say p_4, is interior to the triangle determined by the other three. We first show that the set E of points of Ω which lie on the line segment p_1p_2 is dense in p_1p_2. For this purpose it is enough to show that for each pair of distinct points x, y in E there exists a point z in E distinct from x and y and lying between x and y. Suppose x to be between p_1 and y. Construct lines p_1p_4 and p_2p_4, and let q_1 and q_2 be their intersections with p_2p_3 and p_3p_1 respectively. Construct lines xq_1 and yq_2, and let w be their intersection. Let z be the intersection of the lines p_3w and p_1p_2. Clearly z is the required point.

Similarly it follows that the set of points of Ω which lie on the segments p_2p_3 and p_3p_1 is dense in these segments. Ω contains points of intersection of lines joining points of these dense sets, and hence Ω must be dense in the plane.

Case 2. The four points form a convex quadrilateral. Since this quadrilateral is not a parallelogram we may extend a pair of opposite sides to a point of intersection, thus obtaining a triangle. Since the intersection of the diagonals of the quadrilateral lies inside this triangle we may now apply the argument for *Case* 1.

The necessity of conditions (i) and (ii) is easily verified.

If the plane referred to is the real projective plane, the condition(ii) may be discarded.

Also solved by Robert Breusch, Harley Flanders, J. K. Jacobsen and C. D. Gorman, and P. G. Kirmser.

Series Representing an Irrational Number

4627 [1955, 127]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

In Chrystal, *Textbook of Algebra*, vol. 2, p. 225, there is the following theorem on the representation of an irrational number:

The number represented by the series

$$\sum_{n=1}^{\infty} \frac{p_n}{r_1 r_2 \cdots r_n}$$

is irrational provided that

(1) r_n and p_n are integers such that $0 < p_n < r_n$,

(2) $r_{n+1} > r_n > 1$,

(3) the sequence $\{r_1 r_2 \cdots r_n\}$ includes all powers of the primes.

(A) Construct a counter-example, *i.e.*, a number of the stated form which is rational even though conditions (1), (2) and (3) are satisfied.

(B) Complete the list of conditions (1), (2) and (3) so that the theorem is indeed valid.

Solution by Ivan Niven, University of Oregon. (A) The following number is of the required form and is rational:

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n!} - \sum_{n=1}^{\infty} \frac{1}{(n+1)!} = 1.$$

(B) The theorem is valid if the following condition is also satisfied:

(4) $p_n < r_n - 1$ for infinitely many values of n.

For if the series converged to a rational number, a/b, then by condition (3) we could choose an integer m so that b is a divisor of $r_1 r_2 \cdots r_m$. When we multiply the equation

$$\frac{a}{b} = \sum_{n=1}^{\infty} \frac{p_n}{r_1 r_2 \cdots r_n}$$

by $r_1 r_2 \cdots r_m$, and rearrange terms to get

$$a \frac{r_1 r_2 \cdots r_m}{b} - \sum_{n=1}^{m} \frac{p_n r_1 r_2 \cdots r_m}{r_1 r_2 \cdots r_n} = \sum_{i=1}^{\infty} \frac{p_{m+i}}{r_{m+1} r_{m+2} \cdots r_{m+i}},$$

we note that the left side is an integer. We get a contradiction by establishing that the series on the right converges to a value between 0 and 1. Indeed, the series on the right is positive because of condition (1), and by condition (4) the series is less than

$$\sum_{i=1}^{\infty} \frac{r_{m+i} - 1}{r_{m+1} r_{m+2} \cdots r_{m+i}},$$

but this series converges to the value 1 because the sum of the first k terms has the value

$$1 - \frac{1}{r_{m+1} r_{m+2} \cdots r_{m+k}}.$$

Thus the proof is complete and we make the following comment. Condition (2) may be replaced by the weaker inequality $r_n > 1$, and the theorem remains valid by the same proof.

Also solved by Robert Breusch, Leonard Carlitz, S. W. Golomb, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Beginning Algebra for College Students. By Lloyd L. Lowenstein, New York, John Wiley and Sons, Inc., 1953. xiii+279 pages.

This book is designed for students who have no previous preparation in algebra but who are mature enough to profit by a postulational approach to the subject. Since this approach is the distinctive feature of the book, most of the review will be given over to it.

The seven postulates on which the rest of the book is based refer to the set of natural numbers: this set is closed under addition and multiplication, which are also associative and commutative, and the distributive law holds. Each postulate is introduced by examples from arithmetic and some are followed by generous discussions which include examples of sets or operations for which the stated property does not hold. Repeated application of the postulates leads to generalizations in the form of rules, such as the laws of exponents and certain rules for factoring.

The algebra of the natural numbers is developed somewhat (parentheses, exponents, polynomials, simple equations) before introducing any new operations. Extension of the number system is motivated by inverting the operations of addition and multiplication and devising an enlarged system in which the seven postulates continue to hold. The natural numbers are introduced on page 4, zero on page 43, directed numbers on page 52, and fractions on page 88. The rational number system is defined on page 125, and the property of closure with respect to the four elementary operations is shown.

In the second half of the book, pages 137 to 248, some of the topics begun earlier receive enlarged treatment, *e.g.*, polynomials, factoring, equations, and exponents, while new topics include functions and graphs, the real number system (with a discussion of radicals), and the complex number system.

The placing of the irrational and complex numbers at the end is good in a book which is not just a review of high school algebra. Another good feature is the six-page appendix containing discussions of questions asked in the text, together with references for further study. In general the exposition throughout is clear, and the problems are well chosen.

As for the distinctive feature of the book, the postulates are carefully stated, and they are actually used to derive rules of procedure. The definitions are also carefully stated, and many are preceded by a generous preliminary discussion. (An exception is the word "term," which is not defined at all.) However, the same care is not used in extending to the gradually enlarged number system the properties stated in the postulates. It is not made clear that these properties require proof based on the original postulates and such definitions as have been made or

may be made. Thus on page 44, after the definition of zero, we find the declaration that all the rules previously built up are going to apply to the new number. Then the commutative, associative, and distributive laws are stated with zero as one element.

On the whole the book accomplishes well its main purpose, which is to present algebra as an orderly development from a few basic assumptions. After studying it, a student should be quite ready for college algebra.

W. G. CLARK
Mt. Union College

Algebra for College Students. By W. M. Whyburn and P. H. Daus. New York, Prentice Hall, Inc. 1955. 254 pp. plus tables and index.

As the authors remark in the preface, the "difficulties experienced by college mathematics students result not only from weaknesses in the manipulative skills of algebra, but also from deficiencies in geometry and arithmetic and the lack of correct habits of thinking and reasoning." These incorrect habits are often inherited by the students from the teachers in high school mathematics. The reviewer recalls a remark of a high school teacher in one of his classes, "I do not see what logic has to do with mathematics."

Texts like the Chicago syllabus or *Principles of Mathematics* by Allendoerfer and Oakley (McGraw-Hill) are specially aimed at this deficiency. The present text is a compromise between such texts and the usual college algebra texts which are often "an unenthusiastic rehash" of high school mathematics.

The book has eight chapters and an introductory review of elementary arithmetic. Chapters I–II are devoted to direct and inverse operations on numbers. The fundamental properties are given by axioms and the usual "rules" such as those concerning removal of parentheses, multiplying two negative numbers *etc.* are derived from these axioms. Real numbers are introduced by the *axiom of linear measure.* Article 16 on "Change of Base" brings out clearly the role of number 10 in writing natural numbers in arithmetic. The chapters conclude with factor theorem and Euclidean algorithm for finding the H.C. F.

Chapter III entitled "Fractions" deals with negative exponents, terminating and non-terminating decimals, approximation, variation illustrated by simple interest and simple discount, and elements of the trigonometry of an acute angle. Chapter IV treats of linear equations, graphically as well as arithmetically. The latter method introduces us to determinants and matrices. Chapters V–VI treat quadratic equations. Complex numbers are introduced and their sum and product properly defined. Simple conics come in as graphs of simple quadratics in two variables. The sine law and cosine law complete the trigonometry of a triangle. Chapter VII treats of exponents and logarithms. Irrational exponents are introduced by two explicit assumptions. The last chapter deals with "Sequences of Numbers" and "Mathematical Induction." The latter is stated in an axiom underlying a procedure used to prove or disprove "conjectured" results of a certain type.

The text is very carefully written and does not contain anything that a student will have to forget when he learns more mathematics. There are no bold-type framed "Formulas" nor any annoying schemes for working problems: Step I, Step II, *etc.* such as we find in many texts.

In conclusion, a few minor defects may be noted here: On page 9, a variable is defined as "a quantity that may change its value during the discussion." On page 113, the x and y intercepts are defined as "points"; and almost immediately we read "If the intercepts are small, an additional point is desirable."

V. D. GOKHALE
Atlanta University

Parmi les belles figures de la Géométrie dans l'espace (*Géométrie du tétraèdre*). By Victor Thébault. Paris, Vuibert, 1955. 11+287 pages. 2000 french francs.

Victor Thébault is known on both sides of the Atlantic as a geometer interested in the triangle and the tetrahedron. For over thirty years he has been seeking analogies between the geometry of the triangle and the geometry of the tetrahedron. His researches have been published in many journals including this MONTHLY and have been issued in the form of memoirs, articles, notes and problems. He has now gathered into one volume the principal results concerning the tetrahedron, its associated points, lines, planes and spheres.

After an introduction explaining the terminology and notation, which includes a number of distance relations, the material is divided into four chapters: Chapter I: basic configurations; Chapter II: spheres associated with a tetrahedron and a skew polygon; Chapter III: additions to the recent geometry of the tetrahedron; Chapter IV: problems proposed concerning the tetrahedron.

The topics in the first chapter include the Monge point, Euler lines, the 12-point sphere, the orthocentroidal and the Miquel spheres, the Steiner ellipsoids and some hyperboloids, associated planes and tetrahedra and some special tetrahedra. In the second chapter is the Longchamps sphere and a skew curve associated with a tetrahedron; it ends with a discussion of spheres associated with a skew polygon whose vertices lie on a sphere. The third chapter treats of the first and second Lemoine points, the spheres of Tucker, Adams, Lucas and Hagge, and the orthopole of a line. In the final chapter are 81 original problems proposed since 1913 whose solutions have led often to interesting results.

This volume should be of interest to those who are acquainted with the geometry of the triangle and the tetrahedron, and to all those who enjoy witnessing a trained mind at work in bringing into view the relations existing among lines, planes, spheres and tetrahedra associated with the configuration formed by four planes.

J. R. MUSSELMAN
Western Reserve University

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

REPRINTS OF VOLUMES OF THE MONTHLY

By arrangement with the Mathematical Association of America, Johnson Reprint Corporation of New York is now reprinting the first twenty volumes of the *American Mathematical Monthly*. These early volumes have not been available in bound form for some time, and the reprint will therefore afford libraries an opportunity to complete their sets. Cloth bound sets are priced at $275.00, paper bound sets are available at $245.00, and individual paper bound copies of Volumes 1–9 are priced at $15.00, and of Volumes 10–20, $10.00. It is anticipated that the reprinted sets will be available by May, 1956.

Orders and inquiries should be addressed to: Johnson Reprint Corporation, 125 East 23 Street, New York 10, New York.

MATHEMATICS INSTITUTE FOR TEACHERS

The second annual Mathematics Institute of the University of Oklahoma will be held June 4–15, 1956, in the air-conditioned Union Building on the main campus.

Opportunity will be given to study methods of teaching the various subjects in the high school mathematics curriculum, to get a deeper insight into the subject matter of high school mathematics with its new-type applications and to acquire source material for use with talented students. A study will also be made of some relatively simple developments in modern mathematics.

One hour of graduate credit will be given for either week of the Institute—two hours for both weeks. Send inquiries to the Institute Chairman, J. O. Hassler, Department of Mathematics, University of Oklahoma, Norman, Oklahoma.

COMMISSION ON MATHEMATICS

A Commission on Mathematics that will investigate the need for revision of the secondary school mathematics curriculum has been appointed by the College Entrance Examination Board.

New frontiers in mathematics are creating almost unlimited opportunities for growth in mathematical knowledge and its applications to physical science and engineering, to the social and biological sciences, and to business and industry. The Commission has been established by the College Board in recognition of a growing divergence between these developments and the type of mathematics taught in secondary school.

After stressing modern mathematical concepts only in graduate courses for many years, many college mathematics departments are readjusting freshman

courses to include these ideas. No equivalent trend has yet developed widely on the high school level.

The Commission on Mathematics, a group of 13 high school and college mathematics teachers, proposes to investigate the need and possibilities for revision of the high school syllabus. One or more conferences of high school and college teachers and administrators will be summoned to advise and assist the Commission. Following such meetings, and in the light of the experience and deliberations of its own members, the Commission will recommend necessary and desirable action to the College Board.

Curricular reform has often failed to result from the work of previous conferences and commissions in mathematical education because their recommendations have been only partially implemented. In view of this, the College Board is prepared to take further action. If there is general agreement as to the need and form of curricular revision, the College Board will, if necessary, foster the development of teaching materials and workshops to prepare teachers to utilize these materials.

With this charge at hand, the Commission met for the first time at Ann Arbor, Michigan, in August, 1955, and reached the following tentative conclusions:

(1) High school mathematics is in need of revision which would include the addition of new materials and the elimination of some and reorganization of other topics now being taught;

(2) In considering a topic for deletion or inclusion the Commission will bear in mind the relevance of the topic to modern mathematics, social and natural science, and engineering;

(3) Liaison with and the cooperation of all groups interested in school and mathematics curricula is essential; and

(4) The development of a single definitive syllabus is undesirable, if not impossible. The Commission therefore proposes to formulate a broad list of topics and to suggest where and how they might be introduced.

Members of the Commission include: Professor A. W. Tucker, Princeton University (Chairman); Professor C. B. Allendoerfer, University of Washington; Mr. E. C. Douglas, The Taft School, Watertown, Connecticut; Professor H. F. Fehr, Teachers College, Columbia University; Miss Martha Hildebrandt, Proviso Township High School, Maywood, Illinois; Dean A. E. Meder, Jr., Rutgers University; Professor Frederick Mosteller, Harvard University; Professor E. P. Northrop, University of Chicago; Dr. E. R. Ranucci, Weequahic High School, Newark, New Jersey; Mr. Robert Rourke, The Kent School, Kent, Connecticut; Professor G. B. Thomas, Jr., Massachusetts Institute of Technology; Professor Henry Van Engen, Iowa State Teachers College; Professor S. S. Wilks, Princeton University.

The Commission welcomes the assistance and suggestions of organizations and individuals. Those interested in corresponding with the Commission should write to Mr. Robert Kalin, Executive Assistant, Commission on Mathematics,

P.O. Box 592, Princeton, New Jersey.

NEW TELEVISION SERIES

Professor Francis A. C. Sevier of the College of South Jersey is presenting a television series on WFIL-TV. The series began on January 30 and will run through May 14. It is titled "Mathematics in Modern Life." Outlines of Professor Sevier's syllabus can be obtained by writing to him at 406 Penn Street, Camden 2, New Jersey.

WAYNE UNIVERSITY COMPUTATION LABORATORY PROGRAM

The Computation Laboratory of Wayne University is again offering an intensive training program of three weekly courses.

First Course: July 23–28. Automatic computers—their application and evaluation.

Second Course: July 30–August 4. Electronic data processing in business and government.

Third Course: August 6–11. Applications of computors to engineering, science and industry.

The final program with full information on lectures, staff, and other pertinent information will be distributed well in advance of the summer courses. For further information, write to A. W. Jacobson, Director, Computation Laboratory, Detroit 1, Michigan.

DU PONT FELLOWSHIPS AT THE UNIVERSITY OF CHICAGO

The University of Chicago announces six graduate fellowships of $1,920 for students who wish to prepare for teaching chemistry, mathematics or physics in secondary schools. E. I. du Pont de Nemours and Company has granted the University funds for these fellowships to encourage able college graduates of both sexes to enter high-school mathematics and science teaching.

The du Pont fellows will register for science and/or mathematics courses, for professional courses in the teaching of the subject, and for apprentice teaching courses. Their program of study is designed to enable them to meet the requirements for the secondary-school teaching certificate issued by most states and to advance them toward a Master of Science degree.

Applications for fellowships for 1956–1957 must be made before May 15, 1956. Further information about the du Pont Fellowships may be obtained from the Center for Teacher Education, University of Chicago, Chicago 37, Illinois.

INSTITUTE OF MATHEMATICAL SCIENCES TEMPORARY MEMBERSHIPS

The Institute of Mathematical Sciences of New York University, Professor R. Courant, Director, offers *temporary memberships* to mathematicians and other scientists holding the Ph.D. degree who intend to study and do research in the fields in which the Institute is active. These fields include functional analysis, ordinary and partial differential equations, mathematical physics, fluid dy-

namics, electromagnetic theory, numerical analysis and digital computing, and various specialized branches, such as group theory, topological methods of analysis, hydromagnetics, reactor theory.

The temporary members will have complete freedom to select their own activities. They may participate in the advanced graduate courses, research seminars, and research projects of the Institute and will have the opportunity of using computational facilities.

The temporary members will receive a stipend commensurate with their status. Membership will be awarded for one year, but may be renewed. Special arrangements can be made for applicants who expect to be on leave of absence from their institutions.

Requests for information and for application blanks should be addressed to the Membership Committee, Institute of Mathematical Sciences, 25 Waverly Place, New York 3, New York.

INDUSTRIAL MATHEMATICS SOCIETY

Mathematics in an Industrial Economy is a new booklet issued by the Industrial Mathematics Society. It will be sent free to all who request it by writing to the Industrial Mathematics Society, 100 Farnsworth, Detroit 1, Michigan.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1956.

Boston University. July 9 to August 18: Professor Scheid, calculus of variations.

The Catholic University of America. June 27 to August 10: Mr. Tomiuk, advanced calculus I; Professor Moller, higher algebra I; Professor Wiegmann, introduction to matrix theory; Professor Rice, advanced calculus II; Professor Finan, higher algebra II; Professor Ramler, college geometry, synthetic projective geometry, differential equations.

Columbia University, Teachers College. July 9 to August 17: Professor Fehr, teaching arithmetic in elementary school, current problems in teaching secondary school mathematics; Professors Fehr and Rosskopf, research in teaching mathematics, departmental seminar in teaching mathematics; Professor Rosskopf, teaching of algebra in secondary school, history of mathematics; Dr. Sobel, field work in mathematics, teaching non-academic mathematics in the high school; Mr. Rourke, survey of higher mathematics for teachers, professionalized subject matter in advanced secondary school mathematics; Professors Fehr, Rosskopf, Sobel and Rourke, conferences and discussions on current questions in the teaching of mathematics.

DePaul University. June to August: Professor DeCicco, classical Lie theory of continuous groups, fundamental concepts of algebra; Professor Caton, introduction to the calculus of variations. June to August (evenings): Professor Caton, engineering mathematics.

Duke University. June 12 to June 17: Professor Carlitz, probability; Professor Roberts, plane geometry and trigonometry from the advanced standpoint; Professor Thomas, mathematical analysis for teachers; Professors Carlitz, Roberts and Thomas, thesis seminar.

Indiana University. June 14 to August 10: Professor Wolfe, theory and application of statistics, projective geometry; visiting Professor Snapper (Miami University), topics for mathematics teachers, higher algebra I; Professor MacKenzie, differential equations, theory of functions of a complex variable I. For the first time there will be available for qualified secondary school teachers three David A. Rothrock Summer Scholarships amounting to $200 plus basic fees.

North Carolina State College. Southern Regional Graduate Summer Session in Statistics. June 11 to July 20: Professor Hader, statistical methods I; Professor Cox, statistical methods II; Dr. Grandage, theory I; Professor Duncan (University of Florida), theory II; Professor Finkner, sample survey designs; Staff, special problems; Professor Anderson, econometric methods; Professor Monroe, advanced topics in statistical analysis II; Professor Harrell, introduction to linear programming; Professor Hildreth (Michigan State University), linear equations; Professor Heady (Iowa State College), estimation and analysis of production functions; Professor Levine, advanced calculus; Professor Smith (University of North Carolina), stochastic processes and their applications.

The Institute in Quantitative Research Methods in Agricultural Economics, sponsored by the Social Science Research Council, will meet at the College in conjunction with the Graduate Summer Session in Statistics.

Northwestern University. June 23 to August 18: Engineering mathematics I, theory of statistics, introduction to the theory of numbers, the history of mathematics II, foundations of calculus for teachers, topics in modern mathematics for teachers, nonlinear problems of the physical sciences, introduction to the theory of groups.

Syracuse University. July 2 to August 10: Professor Cole, analysis and applications I; Professor Gilbert, introduction to probability; Professor Kibbey, intermediate course in algebra II; Professor Exner, heuristic methods in elementary geometry. August 13 to September 14: Professor Hemmingsen, analysis and applications II (vector analysis); Professor Welch, higher mathematics for engineers and scientists I.

University of Buffalo. July 2 to August 10: Professor Montague, rings and ideals; Professor Gehman, theory of games.

University of California. June 18 to July 28 and July 30 to September 8. Graduate Courses in Statistics. The faculty will include Professor Cox (University of North Carolina), Professor Bates (Mount Holyoke College), Professor Blackwell and Mr. Ferguson of the Department of Statistics. The program includes two of the usual undergraduate courses in each session, adapted primarily to meet the need of students transferring from other centers who would like to undertake advanced study at the University of California during the regular

academic year. Also a graduate seminar will be conducted by Professor Blackwell. This seminar will allow for individual consultation for students working toward higher degrees.

June 18 to July 28. Graduate Course in Mathematics: Professor Jacobson (Yale University), ring theory.

University of Chicago. June 25 to August 31: A program in analysis including special courses and seminars will be offered. Visitors in residence will include A. P. Calderon, H. Helson, J. E. Littlewood, W. Rudin, R. Salem, and A. C. Schaeffer. Further information can be obtained from E. H. Spanier, Department of Mathematics, University of Chicago, Chicago 37, Illinois.

University of Colorado. June 18 to August 25: Professor Whyburn, theory of functions; Professor Hultquist, vector analysis; Dr. Crow, mathematical statistics; Professor Britton, complex variables, boundary value problems.

University of Delaware. June 18 to July 27: Professor Webber, fundamental concepts of mathematics I (designed especially for teachers).

University of Florida. June 15 to August 13: Professor Blake, introduction to mathematical thought; Professor South, mathematical statistics; Professors Lang, Blake and Smith, higher mathematics for engineers and physicists; Professor Smith, tensor analysis; Professor Phipps, foundation of geometry; Professor Cowan, Fourier series; Professor Gager, history of elementary mathematics; Staff, individual work.

University of Illinois. June 18 to August 11: Professor Suzuki, linear algebra; Professor Landin, number, length, area; Professor Bateman, measure and integration.

University of Maryland. June 25 to August 3: Professor Jackson, foundations of number theory (primarily for teachers); Professor Hall, elementary topology; Dr. Rosen, introduction to complex variable theory.

University of Michigan. June 25 to August 17: Professor Alexander, theory of probability, elementary mathematical statistics; Professor Brauer, higher algebra, seminar in algebra; Professor Carver, mathematical statistics, finite differences; Professor Coburn, operational mathematics; Professor Craig, significance tests; Professor Dushnik, advanced calculus; Dr. Gehring, advanced mathematics for engineers; Professor Gentry, synthetic projective geometry, topics in geometry; Professor Hay, vector analysis, theory of plates and shells; Professor Herzog, functions of a complex variable, partial differential equations; Professor Hildebrandt, calculus of variations; Professor Jones, teaching of algebra, history of geometry; Professor LeVeque, matrix theory; Professor Ullman, Fourier series, application of functions of a complex variable; Professor Young, foundations of mathematics, general spaces.

A Summer Institute for Teachers of Collegiate Mathematics will be held at the University during the summer session. (See this MONTHLY, March, 1956.)

University of Nebraska. June 13 to August 3: Professor Jackson, differential equations; Professor Magnus, vector analysis; Professor Leavitt, topics in algebra; Staff, reading course, thesis.

University of North Carolina. June 7 to July 14: Professor Brauer, some recent results in number theory; Professor Cameron, fundamental concepts; Professor Garner, history of mathematics; Professor Jones, non-euclidean geometry; Professor Linker, differential equations; Professor Mackie, theory of equations; Professor Winsor, introduction to higher geometry. July 16 to August 22: Professor Hill, elementary mathematical statistics; Professor Hoyle, elementary algebra from an advanced viewpoint; Professor Lasley, analytic geometry from a higher standpoint; Professor MacNerney, summability; Professor Wall, introduction to the theory of determinants and matrices.

University of Oklahoma. June 7 to August 5: Professor Bernhart, college geometry; Professor Pan, elementary differential equations; Professor LaFon, vector analysis; Professor Springer, ordinary and partial differential equations; Professor Brixey, theory of numbers; Professor Goffman, theory of probability.

University of Pennsylvania. June 25 to August 4: Professor Herstein, matrix theory and numerical applications; Professor Epstein, methods of analysis; Professor Fine, probability and statistics; Professor Gottschalk, concepts of mathematics (this course carries credit towards a master's degree in education with specialty in mathematics).

University of South Carolina. Professor Lytle, theory of equations; Professor Novak, college geometry; Professor Hedberg, theory of algebraic numbers; Professor Strebe, theory of functions of a real variable; Professor Williams, theory of functions of a complex variable.

University of Texas. June 5 to July 17: Professor Craig, vector and tensor analysis, applications of tensor analysis; Professor Moore, theory of sets; Professor Guy, potential theory (two terms). July 18 to August 31: Professor Cooper, functions of a complex variable; Professor Lubben, modern algebra; Professor Wall, infinite processes, analytic functions; Professor Guy, potential theory.

University of Washington. Professor Avann, linear algebra; Professor Cramlet, differential equations; Professor Ballantine, vector analysis; Professor McFarlan, topics in applied analysis; Professor Leipnik, advanced analytic geometry; Professor Haller, differential geometry; Professor Livingston, convolution transforms.

University of Wisconsin. June 25 to August 17: Professor Evans, vector analysis; Professor Hammer, theory and operation of computing machines; Professor Immel, introduction to mathematical statistics, survey of the foundations of arithmetic; Professor Jones (University of North Carolina), topics in geometry, topics in topology; Professor Kaplansky (University of Chicago), determinants and matrices, advanced topics in algebra; Professor Marden, applied differential equations, potential theory; Professor Mayor, projective geometry.

University of Wyoming. June 11 to July 13: Professor Varineau, theory of equations; Professor Steen, ordinary differential equations; Professor Barr, college geometry; Professor S. R. Smith, numerical analysis. July 16 to August

17: Professor Schwid, partial differential equations, Fourier series and boundary value problems, theory of functions of a complex variable; Professor Neubauer, history of mathematics; Professor W. N. Smith, mathematical theory of probability.

West Virginia University. June 6 to July 17: Professor Cunningham, Fourier series and partial differential equations; Professor Posey, advanced calculus; Professor Stewart, history of mathematics. July 18 to August 24: Mr. Cochran, modern geometry; Professor Peters, advanced calculus; Professor Vest, operational methods in partial differential equations.

PERSONAL ITEMS

Assistant Professor H. L. Alder of the University of California, Davis, has been promoted to an associate professorship.

Mr. A. I. Benson, formerly a staff member of the Los Alamos Scientific Laboratory, New Mexico, is manager of Lynn Digital Computations, General Electric Company, West Lynn, Massachusetts.

Mrs. Audrey J. Benson, previously a research assistant at the Los Alamos Scientific Laboratory, New Mexico, is now an assistant mathematician at the Rand Corporation, Lincoln Laboratory, Lexington, Massachusetts.

Mr. Jonas Beraru, formerly a mathematician with the Reeves Instrument Corporation, New York, New York, has a position as a member of the technical staff of Hughes Aircraft Company, Culver City, California.

Miss Ida Brisky has a position as a systems engineer with the Datamatic Corporation, Newton Highland, Massachusetts.

Mr. C. L. Davis, previously an instructor at the General Motors Institute, has accepted the position of Mathematician in the Mathematics and Computing Group of Curtiss-Wright Research Division, Clifton, New Jersey.

Mr. M. J. Di Carlo-Cottone, formerly a mathematician for the United States Air Force, Washington, D. C., is a member of the technical staff of Bell Telephone Laboratories, Whippany, New Jersey.

Dr. D. O. Ellis, associate mathematician for the Rand Corporation, Santa Monica, California, has accepted a position as Senior Research Scientist with the National Cash Register Company, Electronics Division, Hawthorne, California.

Mr. J. H. Fountain is employed as a research engineer by Consolidated-Vultee Aircraft Corporation, San Diego, California.

Mr. M. P. Friedman of Brooklyn College is now with the Research Division of the College of Engineering, New York University.

Mr. F. W. Gibson, formerly a mathematical analyst with Lockheed Aircraft Corporation, Burbank, California, is now a research engineer for the Radioplane Company, Van Nuys, California.

Dr. G. R. Glabe has been appointed to an assistant professorship at Sacramento State College.

Visiting Professor G. H. Handelman of Rensselaer Polytechnic Institute is in charge of a new program of research sponsored by the Air Force Office of Scientific Research, Air Research and Development Command, Baltimore.

Mr. M. A. Kirchberg, previously a teacher at Hopkins Township School, Michigan, has a position as a computer operator for Allis-Chalmers Manufacturing Company, Milwaukee, Wisconsin.

Miss Lola F. Kiser of the University of Georgia has been appointed to an assistant professorship at Birmingham-Southern College.

Professor Emeritus E. E. Moots of Cornell College has been appointed to a visiting professorship at Whittier College for the second semester of 1955–1956.

Assistant Professor Eugene Park of Clemson College has been promoted to an associate professorship.

Mr. Conrad Rennemann, Jr., formerly an aeronautical research scientist with the National Advisory Committee for Aeronautics, Langley Field, Virginia, has a position as a senior aerodynamicist with the Republic Aviation Corporation, Farmingdale, L. I., New York.

Mr. L. A. Rife of Mississippi Southern College has been promoted to an assistant professorship.

Mr. J. J. Schoderbek, previously a research assistant at the University of Michigan, Willow Run Research Center, Ypsilanti, Michigan, has a position as a research engineer for Lockheed Aircraft Corporation, Van Nuys, California.

Mr. C. E. Seabold, recently a mathematician at Wright-Patterson Air Force Base, Ohio, is employed as a senior technical programmer with the Standard Oil Company of Ohio.

Mr. H. F. Simmons, formerly a graduate assistant at Iowa State College, has been appointed to an assistant professorship at the University of Wichita.

Mrs. Lois B. Smith is teaching in the San Diego City Schools, California.

Mr. A. G. Wilford, previously a meteorologist for the Department of Transport, Toronto, Canada, is employed as a programmer by Computing Devices of Canada, Ottawa, Canada.

Dr. John Wishart, director of the Statistical Laboratory at St. Andrew's Hill, Cambridge, England, will be a visiting professor at the University of Wisconsin for the first semester of the academic year 1956–1957.

Associate Professor Herbert Wolf of Tri-State College has a position as a research mathematician at the Avco Manufacturing Company, Stratford, Connecticut.

Mr. T. I. Gilroy, fellow at the University of Wisconsin, died in July, 1955.

Miss Clara Hancock, retired instructor of Junior College, Virginia, Minnesota, died on March 23, 1955. She was a member of the Association for thirty-five years.

Mr. D. H. Leavens of Colorado Springs, Colorado, died on December 3, 1955. He was a charter member of the Association.

Professor Emeritus H. P. Manning of Brown University died on January 11, 1956. He was a charter member of the Association.

Professor Emeritus F. R. Morris of Fresno State College died on January 6, 1956. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1955

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1955. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

As was the case at the end of the preceding year, the total funds of the Association at the end of the year 1955 include balances held for the Committee on Visiting Lecturers and the Committee on the Undergraduate Program. These funds will be expended as the plans of these Committees require.

The Current Fund ended the year with a surplus of $2,065. Anticipated expenditures during 1956 indicate that a rather substantial deficit will be incurred in 1956. The Finance Committee is considering whether an increase in dues to $5.00 per year may become necessary. The Secretary-Treasurer will be glad to have the comments of members of the Association on this matter.

I. Total Funds of the Association on January 1, 1955

M & T Trust Co., Buffalo	$ 11,467.06	Current Fund	$ 495.62
Securities	104,831.00	Carus Fund	19,626.04
		Chace Fund	14,797.47
		Houck Fund	10,830.03
		Chauvenet Fund	1,363.14
		Dunkel Fund	17,865.41
		General Fund	40,348.91
			105,326.62
		Visiting Lecturers Fund	10,462.28
		Fund for Com. on Und. Program	509.16
	$116,298.06		$116,298.06

II. Current Fund

Balance, January 1, 1955	$ 495.62	Monthly	
Dues	23,004.76	Publication	$19,794.83
Initiation fees	908.00	Reprints	510.87
Subscriptions	4,688.13	Editor's Office	1,986.34
Sale of back numbers	260.23	Secretary-Treasurer's Office	
Advertisements	5,993.50	Clerical help	8,805.91
Sale of exchange periodicals	125.50	Postage and printing	2,037.70
Interest on General Fund	1,493.97	Office expenses	405.90
Income from Hardy Fund	120.00	Bank and Auditing fees	238.55
Transfer from other Funds	2,323.42	Board of Governors	1,737.83
Foreign exchange and bank charges	3.70	Meetings	684.00
		Committees	160.00
		Subventions	493.53
		Balance, December 31, 1955	$ 2,561.37

	III. Carus Fund	IV. Chace Fund	V. Houck Fund
Balance, January 1, 1955	$19,626.04	$14,797.47	$10,830.03
Sales of publications	1,456.41	518.98	7.00
Interest	729.52	547.14	399.69
Increase in value of securities	566.68	425.01	310.47
Less: charges and bank fees	237.71	146.49	51.14
Printing	—	2,302.86	—
Balance, December 31, 1955	$22,140.94	$13,839.25	$11,496.05

	VI. Chauvenet Fund	VII. Dunkel Fund	VIII. General Fund
Balance, January 1, 1955	$ 1,363.14	$17,865.41	$40,348.91
Interest	50.45	659.67	—
Increase in value of securities	39.19	512.42	1,160.49
Less: charges and bank fees	6.37	83.26	—
Balance, December 31, 1955	$ 1,446.41	$18,954.24	$41,509.40

	IX. Visiting Lecturers Fund	X. Fund for Committee on Undergraduate Program
Balance, January 1, 1955	$10,462.28	$ 509.16
National Science Foundation grant	20,500.00	—
Contributions toward travel expenses	1,568.33	—
Ford Foundation grant	—	25,000.00
Less: salaries, services, travel expense	7,535.54	2,313.68
Committee expenses	1,082.54	256.15
Administrative expenses	750.00	1,375.00
Balance, December 31, 1955	$23,162.53	$21,564.33

XI. Total Funds of the Association on December 31, 1955

Current Fund	$ 2,561.37	M & T Trust Co., Buffalo	$ 12,842.52
Carus Fund	22,140.94	Buffalo Savings Bank	20,000.00
Chace Fund	13,839.25	Securities	123,832.00
Houck Fund	11,496.05		
Chauvenet Fund	1,446.41		
Dunkel Fund	18,954.24		
General Fund	41,509.40		
	111,947.66		
Visiting Lecturers Fund	23,162.53		
Fund for Committee on Und. Program	21,564.33		
	$156,674.52		$156,674.52

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 150 persons have been elected to membership by the Board of Governors on applications duly certified.

Angela K. Aivaliotis, Student, Kent State University.

O. H. Alisbah, Ph.D. (Berlin) Professor, University of Ankara, Turkey; Visiting Professor, University of Pennsylvania and Rutgers University.

D. R. Anderson, Student, Bradley University.

Richard E. Anderson, M.S. (Michigan) Grad. Student, University of Michigan.

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THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

A meeting of members of the Association living in the New England Region was held at the University of New Hampshire, Durham, on November 26, 1955, for the purpose of organizing a section of the Association. Professors Howard Eves and Donald Kearns presided at the morning and afternoon sessions, respectively. There were 80 persons in attendance including 56 members of the Association.

At a business meeting with Professor A. A. Bennett as temporary chairman, a petition to the national organization for the establishment of a section was circulated and a set of by-laws was adopted. It was agreed that the name of the section should be the Northeastern Section, even though its geographical boundaries coincide with those of the New England Region. It was felt that this name was more appropriate in view of the fact that four Canadian provinces are to be included in the Section. The following officers were elected for a one-year term: Chairman, Professor Howard Eves, University of Maine; Vice-Chairman, Reverend S. J. Bezuszka, Boston College; Secretary-Treasurer, Professor R. E. Johnson, Smith College.

President E. L. Johnson of the University of New Hampshire gave a short welcoming address at the morning session. There was a tea at the President's home following the afternoon session.

The following papers were presented:

1. *A system of dimensional analysis*, by Reverend S. J. Bezuszka, Boston College.

2. *Concerning sines and cosines*, by Professor Ralph Beatley, Harvard University.

The speaker discussed the use of the law of sines to solve triangle problems normally solved by the law of cosines, and *vice versa*.

3. *The structure of a ring*, by Professor R. E. Johnson, Smith College.

There are six basic operations that can be performed on a given set of rings to yield new rings. It was shown how these operations could be used to express each semi-prime ring in terms of prime rings.

4. *Some applications of matrices*, by Professor R. A. Rosenbaum, Wesleyan University.

The incorporation into the undergraduate curriculum of diverse applications of matrices was recommended for a number of reasons, chief among them being the stimulus to a development of mathematical maturity on the part of the student. Examples were drawn from the areas of recursion relations (Fibonacci sequence, and the problem of the weighted string) and geometry (Möbius tetrads). In the latter case the symbiotic relationship of algebra and geometry was emphasized.

5. *Mathematicians at Ticonderoga*, by Professor D. J. Struik, Massachusetts Institute of Technology.

Among the combatants around or at Fort Carillon (Ticonderoga), 1757–58, were men with good mathematical training. Discussed are the cartographers J. F. V. Des Barres and S. Holland, the astronomer-engineer M. C. De Lotbinière and the mathematician-explorer L. A. De Bougainville. De Bougainville's two tome *Calcul integral* (1754, 1756) allows insight into early stages of the theory of elliptic integrals and ordinary differential equations. The work of the cartographers illustrates the importance of the three body problem for longitude determination.

6. *The use of computers in industry*, by Dr. R. F. Clippinger, Raytheon Manufacturing Company, Waltham, Massachusetts, introduced by the Secretary.

R. E. JOHNSON, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Catholic University of America, Washington, D. C., on December 3, 1955. Professor F. E. Johnston, Chairman of the Section, presided at the morning and afternoon sessions. There were 124 persons in attendance, including 96 members of the Association.

By a unanimous vote of the members present, the By-Laws of the Section were amended in such a way as to add a Treasurer to the roster of officers of the

Section. The Secretary presented a report of the Meeting of Section Officers held at Ann Arbor, Michigan, on August 30, 1955. Mr. W. H. Norris, Chairman of the Committee on High School Contests, discussed the results of the 1955 High School Contest conducted by the Section.

The following papers were presented:

1. *Evaluation of the definite integral* $\int_0^\infty u^{-1} \exp(-\phi^2 u/4 - u^{-1})du$, by Professor Anne E. Scheerer, Georgetown University, introduced by Professor Oliphant.

Differentiation of the integral with respect to the parameter ϕ produces a Bessel equation whose solution is a linear combination of Bessel functions of the first and third kinds. The known asymptotic behavior of these functions permits the coefficients to be calculated and the integral thus evaluated.

2. *Generalized complex numbers and addition formulas*, by Professors John Tyler (ret.) and J. A. Tierney, United States Naval Academy, presented by Professor Tierney.

A generalized complex number was defined and a method for writing such a number in polar form was developed. When the modulus is unity the components satisfy addition formulas which are generalizations of the hyperbolic and trigonometric addition formulas. An application to Diophantine analysis was presented.

3. *Two theorems of plane geometry suggested by certain formulae in kinematics*, by Professor J. P. Hoyt, United States Naval Academy.

If v_x and v_y are the rectangular components of the velocity of a point having plane motion, and if a_x and a_y are the rectangular components of its acceleration a, then the rectangular components of a along the tangent to the path and normal to the path are given respectively by $(v_x a_x + v_y a_y)/v$ and $(v_x a_y - v_y a_x)/v$, where v is the magnitude of the velocity. The geometric interpretation of these results leads to two interesting and perhaps new theorems in plane geometry.

4. *Numerical solution of ordinary linear differential equations*, by Dr. C. H. Frick, U. S. Naval Proving Ground.

By use of an integration formula $y_b - y_a = L(y_b', y_b'', \cdots, y_a', y_a'', \cdots)$, where L is a linear function, the numerical solution of a system of linear differential equations was reduced to the solution of a system of linear algebraic equations. Such integration formulas include: the trapezoidal rule, a truncation of the Euler-Maclaurin formula, an extremely accurate formula given in Milne's book on the numerical solution of differential equations, and extensions of the "Milne" formula. The truncated Euler-Maclaurin formula was used in an undergraduate paper by Miss Mary Moskos at Mary Washington College in 1953. One extension of the "Milne" formula was shown.

5. *An approximate solution of a system of non-linear equations*, by Mr. C. H. Murphy, Jr., Aberdeen Proving Ground.

Although the techniques of non-linear mechanics are usually applied to second order equations in one variable, they may be generalized to larger systems of equations. In exterior ballistics, for example, the yawing motion of a spinning projectile acted upon by a non-linear moment is described by two second order non-linear equations in two dependent variables. An approximate solution of these equations is obtained by a perturbation method and applied to the results of free flight tests. The great value of this technique is shown by the excellent internal agreement as well as by good agreement with direct wind tunnel measurements.

6. *Weakly compact transformations*, by Professor J. W. Brace, University of Maryland.

In order that all continuous linear transformations from a Banach space X to an arbitrary Banach space Y which have the property of transforming bounded sets into weakly compact sets have the additional property of transforming weakly compact sets into compact sets, it is necessary and sufficient that X have the following property: $\lim_{n\to\infty} x_n^*(x_n)=0$ for every sequence $\{x_n\}$ in X converging weakly to zero and every sequence $\{x_n^*\}$ in the first adjoint space of X converging *weakly* to zero. The paper also presents Banach spaces which have the desired property.

7. *The evolution of extended decimal approximations to π and e*, by Dr. J. W. Wrench, Jr., David Taylor Model Basin. (By invitation.)

The speaker traced the historical development of extended decimal approximations to π and e from the eighteenth century up to the present and sketched his own work in this field.

R. P. BAILEY, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, April 28, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY, University of Kentucky, Lexington, April 28, 1956.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, U. S. Naval Academy, Annapolis, Maryland, May 5, 1956.

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN

MINNESOTA, Augsburg College, Minneapolis, May 5, 1956.

MISSOURI, Fontbonne College, St. Louis, April 21, 1956.

NEBRASKA, University of Nebraska, Lincoln April 21, 1956.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Oberlin College, Oberlin, April 14, 1956.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA

ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4–5, 1956.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April 20–21, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May 12, 1956.

THE SLAUGHT PAPERS

The Herbert Ellsworth Slaught Memorial Papers are a series of brief expository pamphlets published as supplements to the American Mathematical Monthly. Four numbers have appeared:

1. *Fourier's Series; The Genesis and Evolution of a Theory* by R. E. Langer. v + 86 pages.

2. *Outline of the History of Mathematics (6th edition)* by R. C. Archibald. iv + 73 pages.

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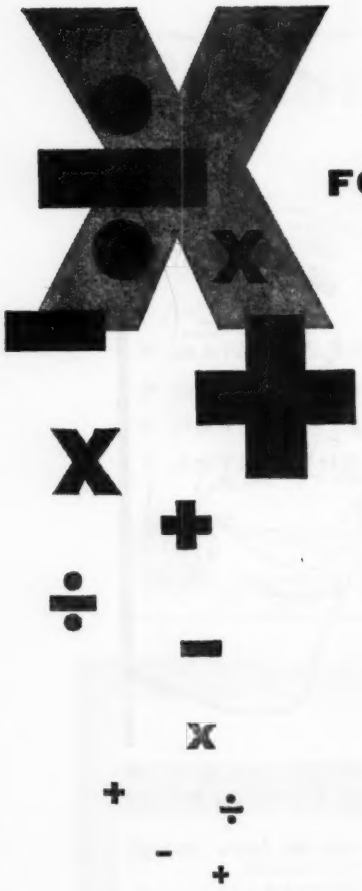

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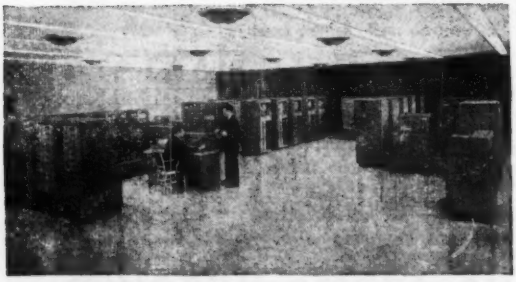

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